BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

14 July 2015

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3.

Examiner 2: Algebra, question A4.

Examiner 3: Analysis, questions B1, B2.

Examiner 4: Analysis, questions B3, B4.

Examiner 5: Applied Mathematics, questions C1, C2, C3, C4.

Examiner 6: Geometry and Topology, questions D1, D2.

Examiner 7: Geometry and Topology, questions D3, D4.

Time allowed: three hours.

Section A: Algebra

A1: Let R be a unital ring. Recall, one characterization of the **Jacobson radical** $J(R)$ is as the set of $x \in R$ such that $1 - axb$ is a unit for all $a, b \in R$.

(a) Let M be a finitely generated non-zero R-module. Let $J(R)M$ be the R-submodule of M consisting of the elements that can be written in the form $a_1x_1 + ... + a_nx_n$ where $a_i \in J(R)$ and $x_i \in M$. Show that $J(R)M$ is strictly contained in M. (Hint: consider a minimal generating set for M.)

(b) Now suppose that R is a finite-dimensional algebra over a field. Directly from part (a) and the above characterization of $J(R)$, show that $J(R)$ is a nilpotent ideal of R.

A2: Let $f(X)$ be an irreducible polynomial over $\mathbb{Q}$ with degree 4. Suppose that $f(X)$ has exactly 2 real roots, α and β. Let E be a the splitting field for $f(X)$ over $\mathbb{Q}$.

(a) Evaluate $[\mathbb{Q}[\alpha] : \mathbb{Q}]$.

(b) Show that $[E : \mathbb{Q}[\alpha, \beta]] = 2$. (Hint: Let γ and δ be the two non-real roots. Explain why $\gamma + \delta$ and $\gamma\delta$ belong to $\mathbb{Q}[\alpha, \beta]$.)

(c) Deduce that $G \cong D_8$ or $G \cong S_4$.

(d) Now suppose that $G \cong D_8$. How many fields K are there such that $\mathbb{Q} \leq K \leq E$?

A3: Let G be the subgroup of $GL_3(\mathbb{C})$ generated by the matrices

$$u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad w = \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}$$

where $\omega = e^{2\pi i/3}$.

(a) Briefly, show that $\{u^iv^j, u^iv^jw, u^iv^jw^2\}$ is a conjugacy class of G except when i and j are both divisible by 3. Hence show that $|G| = 27$ and G has exactly 11 conjugacy classes.

(b) Find the ordinary character table of G.

A4: Let k be a a field and let R denote the polynomial ring $k[x_1, \ldots, x_n]$ in n variables. Recall that a polynomial $f \in R$ is called a **binomial** if its support consists of two monomials. Let I be an ideal in R that is generated by binomials. Show that I has a Gröbner basis consisting of binomials with respect to any monomial order.

Section B: Analysis

B1: Show that the set of rational points on the line is a $\mathcal{G}_{\delta\sigma}$ set but not a $\mathcal{G}_\delta$ set.

B2: Show that the sequence $(\sin nt)_{n=1}^{\infty}$ has no convergent subsequence in the space $C[0,1]$.

B3: Prove that the zeros of the polynomial $p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ all lie in the open disc with center 0 and radius $R = \sqrt{1 + |c_{n-1}|^2 + \cdots + |c_1|^2 + |c_0|^2}$. Assume $p(z) \neq z^n$ to avoid trivialities.

B4: Let $V = \mathbb{C} \setminus [-1,1]$ and $f(z) = z^2 - 1$. Write explicitly a continuous square root g of f on V, show that g actually is continuous on V, and then prove that g is holomorphic on V.

Section C: Applied Mathematics

C1: Using the Green's function method, solve $u''' + u'' = f(x)$ with $u(0) = u'(0) = u''(0) = 0$.

C2: Use the singular perturbation method to obtain a uniform approximate solution to the equation $\varepsilon y'' + y' + y^2 = 0, \quad y(0) = 1/4,\ y(1) = 1/2$, where $0 < \varepsilon \ll 1$ and $0 < t < 1$.

C3: Determine the natural boundary condition at $x = b$ for the variational problem defined by

$$J(y) = \int_a^b L(x, y, y')\,dx + G(y(b)), \quad y \in \mathbb{C}^2[a, b], \quad y(a) = y_0$$

where G is a given differentiable function. As an application of this problem let $L = (y')^2, y \in [0, 1], y(0) = 1$, $y(1)$ is unspecified and $G = y^2$. Find y extremizing this problem.

C4: Transform the problem

$$y'' + xy = 1, \quad y(0) = y(1) = 0$$

to the integral equation

$$y(x) = -\frac{1}{2}x(1 - x) + \int_0^1 G(x, s)s\,y(s)ds$$

where $G(x, s) = x(1 - s)$ when $x < s$ and $G(x, s) = s(1 - x)$ when $x > s$.

Section D: Geometry and Topology

D1: First recall the Hurwitz formula: If $f : X \to Y$ is a non-constant holomorphic map of degree d between two compact Riemann surfaces of genera g_X and g_Y respectively, with ramification divisor R, (which is always an effective divisor), then we have

$$2(g_X - 1) = 2d(g_Y - 1) + \deg R.$$

(a) Prove or disprove that we always have $g_X \geq g_Y$.

(b) When do we have $g_X = g_Y$?

(c) Prove or disprove that any holomorphic map $\phi : \mathbb{P}^n \to E$ is constant, where $\mathbb{P}^n$ is the complex projective n-space and E is an elliptic curve, i.e. genus of E is 1.

D2: Using Čech cohomology techniques, show that, for any meromorphic function ϕ on $\mathbb{C}$, there exist two entire functions f and g such that $\phi = f/g$. (In complex analysis we prove this using the Weierstrass factorization theorem.) Hint: You may need to be reminded that $H^1(\mathbb{C}, \mathcal{O}^*) = 0$.

D3: First note that a **topological group** is a pair (G, μ) where G is a Hausdorff space and μ is a continuous multiplication $G \times G \to G$ which makes G into a group such that the map $g \mapsto g^{-1}$ of $G \to G$ is continuous. Secondly, note that we say a topological group (G, μ) is **compact** if G is compact. Now, let (G, μ) be a compact topological group, X a Hausdorff topological space, and $\Theta : G \times X \to X$ a continuous function such that

1. $\Theta(g, \Theta(h, x)) = \Theta(\mu(g, h), x)$ for all $g, h \in G$ and $x \in X$;
2. $\Theta(e, x) = x$ for all $x \in X$, where e is the identity of G.

For $x \in X$, define

$$G_x = \{\, g \in G \mid \Theta(g, x) = x \,\}$$

as a subgroup of G, define

$$G(x) = \{\, \Theta(g, x) \mid g \in G \,\}$$

as a subspace of X, and define a function α_x from G/G_x to $G(x)$ by $\alpha_x(gG_x) = \Theta(g, x)$. Show that α_x is a homeomorphism for all $x \in X$.

D4: Let X and Y be topological spaces. Then $C(X, Y)$ will denote the set of continuous functions from X to Y with the topology generated by the subbasis

$$\{\, B(K, U) \mid K \text{ is compact in } X \text{ and } U \text{ is open in } Y \,\}$$

where $B(K, U) = \{\, f \in C(X, Y) \mid f(K) \subseteq U \,\}$. Show that $C(I, \mathbb{S}^1)$ is homotopy equivalent to $\mathbb{S}^1$ where I denotes the unit interval $[0, 1]$ in $\mathbb{R}$ and $\mathbb{S}^1$ denotes the unit sphere $\{\, (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \,\}$.

Some Solutions

B3: Note that $R > 1$ under the assumption $p(z) \neq z^n$. If $|z| = R$, then

$$\begin{aligned} |p(z) - z^n| &= \left|c_{n-1}z^{n-1} + \cdots + c_1 z + c_0\right| \\ &\leq \left(|c_{n-1}|^2 + \cdots + |c_1|^2 + |c_0|^2\right)^{1/2} \left(R^{2(n-1)} + \cdots + R^2 + 1\right)^{1/2} \\ &= \left(R^2 - 1\right)^{1/2} \left(\frac{R^{2n} - 1}{R^2 - 1}\right)^{1/2} = \sqrt{R^{2n} - 1} < R^n = |z^n| \end{aligned}$$

by the Cauchy-Schwarz inequality and the sum of a finite geometric series. By the Rouché theorem, $p(z)$ and z^n has the same number of zeros inside the circle $|z| = R$, that is, n. But p is of degree n and has n complex zeros.

B4: For $z \in V$, define $g(z) = \sqrt{|z-1|}\,\sqrt{|z+1|}\,e^{\frac{i}{2}(\arg(z-1)+\arg(z+1))}$, where $-\pi \leq \arg < \pi$. Then $g(z)^2 = |z-1|\,e^{i\arg(z-1)}\,|z+1|\,e^{i\arg(z-1)} = (z-1)(z+1) = f(z)$. Being a composition of continuous functions, g is continuous on $V \setminus (-\infty, -1)$. Letting $x \in (-\infty, -1)$, we have $\lim_{y\to 0^+} g(x+iy) = \sqrt{x^2-1}\,e^{\frac{i}{2}(\pi+\pi)} = -\sqrt{x^2-1}$ while $\lim_{y\to 0^-} g(x+iy) = \sqrt{x^2-1}\,e^{\frac{i}{2}(-\pi-\pi)} = -\sqrt{x^2-1} = g(x)$. Hence g is also continuous across $(-\infty, -1)$.

Further, for $z \in V$ and $z + h \in V$, using the continuity of g on V,

$$\begin{aligned} \lim_{h\to 0} \frac{g(z+h) - g(z)}{h} &= \lim_{h\to 0} \frac{g(z+h)^2 - g(z)^2}{h} \, \frac{1}{g(z+h) + g(z)} \\ &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \, \frac{1}{g(z+h) + g(z)} \\ &= f'(z)\, \frac{1}{2g(z)} = \frac{z}{g(z)} = g'(z). \end{aligned}$$

Since the limit exists, g is holomorphic on V.

D1: Since R is an effective divisor, we always have $\deg R \geq 0$. We also know that the degree of a map is always greater than or equal to 1. We then have

$$2(g_X - 1) = 2d(g_Y - 1) + \deg R \geq 2d(g_Y - 1) \geq 2(g_Y - 1),$$

which forces $g_X \geq g_Y$.

Part (b). Let the common genera of X and Y be g. The Hurwitz formula becomes

$$0 = 2(d-1)(g-1) + \epsilon,$$

where ϵ is the degree of the ramification divisor and we always have $\epsilon \geq 0$. First take $\epsilon = 0$. We then have either $d = 1$ or $g = 1$. The case $d = 1$ is the case when f is an isomorphism and can obviously hold for any g. The case $g = 1$ can hold for any $d \geq 1$ and is known as an isogeny of the elliptic curve involved.

Next take $\epsilon > 0$. In this case we must have $(d-1)(g-1) < 0$, which can hold only when $g = 0$. In that case we have

$$2(d-1) = \epsilon,$$

which holds when $f : \mathbb{P}^1 \to \mathbb{P}^1$ is given, after a change of coordinates, by $z \mapsto z^d$ which ramifies only at 0 and ∞ with ramification indices being d at both points. Hence the ramification divisor becomes $(d-1)\cdot[0]+(d-1)\cdot[\infty]$ whose degree is $2(d-1)$ as our arithmetic predicted.

So when $d = 1$, equality of genera holds for all g. When $d > 1$, equality of genera can hold only if $g = 0$ or $g = 1$.

Part (c). Suppose ϕ is not constant. Then there exist two points $p \neq q$ in $\mathbb{P}^n$ such that $\phi(p) \neq \phi(q)$. Let $\mathbb{P}^1$ be the line in $\mathbb{P}^n$ joining p to q. Restricting ϕ to this line we obtain a non-constant map from a curve of genus 0 to a curve of genus 1, which we have just proved to be impossible. Hence all such ϕ must be constant.

D2: Let $\{U_\alpha\}$ be an open cover of $\mathbb{C}$ with open disks such that we can write

$$\phi|_{U_\alpha} = \frac{f_\alpha}{g_\alpha},$$

where f_α and g_α are holomorphic on U_α and have no common zeros. Define

$$h_{\alpha\beta} = \frac{f_\alpha}{f_\beta}.$$

Since any zero of f_α on $U_\alpha \cap U_\beta$ is a zero of ϕ with the same multiplicity, and since the same is true for f_β, we must have

$$h_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

Moreover if we set $h = \{h_{\alpha\beta}\}$, we see that

$$\delta(h)_{\alpha\beta\gamma} = h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta} = 1,$$

so h defines a cohomology class in $H^1(\mathbb{C}, \mathcal{O}^*)$. Here we use the fact that the covering we chose is Leray; all intersections are contractible. Now since $H^1(\mathbb{C}, \mathcal{O}^*) = 0$, there must be a 0-cochain $k = \{k_\alpha\} \in C^0(\{U_\alpha\}, \mathcal{O}^*)$ such that $\delta(k) = h$. This gives

$$\frac{k_\beta}{k_\alpha} = h_{\alpha\beta} = \frac{f_\alpha}{f_\beta},$$

which in turn gives

$$k_\alpha f_\alpha = k_\beta f_\beta.$$

Thus there exists an entire function F such that

$$F|_{U_\alpha} = k_\alpha f_\alpha.$$

Note that, since each $k_\alpha \in \mathcal{O}^*(U_\alpha)$, the entire function F has the same zeros as ϕ with the same multiplicities.

Similarly there exists an entire function G which has the same zeros of $1/\phi$ with the same multiplicities. Then the function H defined as

$$H = \phi \frac{G}{F}$$

is an entire function which has no zeros. Finally we see that $\phi = \dfrac{FH}{G}$, as claimed.