BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

16 June 2014

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3.

Examiner 2: Analysis, questions B1, B2.

Examiner 3: Applied Mathematics, questions C1, C2, C3, C4.

Examiner 4: Geometry and Topology, questions D1, D2.

Time allowed: three hours.

Section A: Algebra

A1: Let A be an Artinian ring and let B be an Artinian subring of A with $1_B = 1_A$.

(a) Give an example where $J(B) \not\subseteq J(A)$, where J indicates the Jacobson radical.

(b) Give an example where $\ell(B) > \ell(A)$, where ℓ indicates the number of isomorphism classes of simple modules..

(c) Now suppose that $A/J(A)$ is commutative. Show that $J(B) = B \cap J(A)$.

(d) Again suppose that $A/J(A)$ is commutative. Show that $\ell(B) \leq \ell(A)$.

A2: Let L be the splitting field for $(X^2 - 2)(X^3 - 2)$ over $\mathbb{Q}$.

(a) Find the Galois group for L over $\mathbb{Q}$.

(b) How many subfields $K \leq L$ are there such that K is a normal extension field of $\mathbb{Q}$?

A3: Let p and q be primes with $p < q$. Suppose there exists a non-abelian group G with order $|G| = pq$.

(a) Show that $q \equiv 1$ modulo p and that G has a normal Sylow q-subgroup.

(b) What are the degrees of the irreducible $\mathbb{C}G$-characters and how many irreducible characters are there of each degree?

(c) Find the ordinary character table of the nonabelian group with order 21.

Section B: Analysis

B1: Let n, m be integers, dA be the area measure on the open unit disc $\mathbb{D}$, and $\overline{(\ )}$ represent complex conjugation. Evaluate $\dfrac{1}{\pi}\displaystyle\int_{\mathbb{D}} \frac{w^n\,\overline{w}^m}{(1-z\overline{w})^2}\,dA(w)$.

B2: Let f be holomorphic on the open unit disc $\mathbb{D}$ and satisfy $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Suppose that f has two distinct fixed points in $\mathbb{D}$. Prove that $f(z) = z$.

Section C: Applied Mathematics

C1: Prove that every linear second order differential operator with real coefficients is self adjoint provided the weight function $w(x)$ is chosen properly.

C2: Show that a second order Fuchsian differential equation having only two regular singular points is equivalent to a differential equation with constant coefficients, hence solvable in terms of the elementary functions $\sin z$,$\cos z$ and polynomial in z.

C3: Let $J(y) = \int_0^\pi y''^2 dx$ with the constraint $\int_0^\pi y^2 dx = 1$ and with the boundary conditions $y(0) = y''(0) = 0$, $y(\pi) = y''(\pi) = 0$.

(a) Find the function extremizing the functional $J(y)$.

(b) Consider the above problem without the constraint.

C4: Solve the following initial value problem

$$\begin{aligned} u_t = a u_{xx}, \quad (a > 0), t > 0, \quad x \in [0, L], \\ u_x(0, t) = u_x(L, t) = 0, \quad t \geq 0, \\ u(x, 0) = f(x), \quad x \in [0, L] \end{aligned}$$

for functions $f(x)$ such that this initial value problem is well defined.

Section D: Geometry and Topology

D1: Show that every compact subspace of a Hausdorff space is closed.

D2: Let X be a topological space, and let $F(x,t) : X \times [0,1] \to X$ be a homotopy, where $F(x,0) = F(x,1)$ is the identity map $id_X : X \to X$. Prove that for every $x_0 \in X$, the loop $F(x_0,t) : I \to X$ represents an element in the center of $\pi_1(X, x_0)$. (Recall that the center Z of a group G is the set $Z = \{z \in G \,|\, zg = gz \text{ for all } g \in G\}$.)