BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

21 January 2014

Instructions:

- The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

- Hand in separate scripts for each examiner.

  Examiner 1: Algebra, questions A1, A2, A3.

  Examiner 2: Analysis, questions B1, B2.

  Examiner 3: Applied Mathematics, questions C1, C2, C3, C4.

  Examiner 4: Geometry and Topology, questions D1, D2, D3, D4.

Time allowed: three hours.

Section A: Algebra

A1: For each of the following statements, give a proof or a counter-example.

(a) Given a ring R with a unity element and a left ideal I strictly contained in R, then there exists a left ideal J such that $I \subseteq J \subset R$ and J is maximal subject to J being strictly contained in R.

(b) Given a ring R with a unity element, an R-module M and an R-submodule I strictly contained in M, then there exists an R-submodule J such that $I \leq J < M$ and J is maximal subject to J being strictly contained in M.

(c) Given a field F, an F-vector space V and a subspace I strictly contained in V, then there exists a subspace J such that $I \leq J < V$ and J is maximal subject to J being strictly contained in V.

A2: Let $K = \mathbb{Q}[t, i]$ where $t^4 = 2$ and $i^2 = -1$. Let G be the Galois group of $K/\mathbb{Q}$.

(a) Show that $G = \langle \alpha, \beta \rangle$ where α and β are the automorphisms of K such that $\alpha(t) = it$ and $\alpha(i) = i$ and $\beta(t) = t$ and $\beta(i) = -i$.

(b) Find all 6 of the normal subgroups of G. Express them in terms of α and β.

(c) For each of the 6 normal subgroups N, find a polynomial $f_N(X)$ over $\mathbb{Q}$ such that the subfield of K fixed by N is a splitting field for $f_N(X)$.

A3: In this question, you are to construct the ordinary character table of the symmetric group S_4 *by the method indicated.* (Scant credit will be given for recalling some other way of constructing this familiar character table.)

(a) Show that 4 of the irreducible characters χ of the symmetric group S_4 have the form $\mathrm{ind}_H^{S_4}(\psi)$ where $H \leq G$ and ψ is a character with degree $\psi(1) = 1$. For each such χ, find suitable H and ψ, determine the values $\chi(g)$ using the formula for induced characters, then use row orthonomality to check that χ is irreducible. (Do not waste time working out complete character tables of subgroups. You will be using only the characters ψ with degree 1.)

(b) Using those 4 irreducible characters, together with orthogonality properties, complete the character table of S_4.

Section B: Analysis

B1: Consider the metric space (l_∞, d_F), where l_∞ is the set of all bounded real sequences,

$$d_F(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \ .$$

Is this space separable? Is it complete? If it is not complete, find its completion.

B2: Let λ denote the Lebesgue measure on the line, A the set of $x \in \mathbb{R}$ such that the decimal expansion of x does not contain 7. Find $\lambda(A)$.

Section C: Applied Mathematics

C1: Let

$$g(t,x) = \sum_{n=0}^{\infty} t^n \, P_n(x),$$

be the generating function of the Legendre polynomials, $P_n(x)$, with $x \in [-1,1]$. Legendre polynomials satisfy the recursion relations

$$\frac{d}{dx} P_{n+1} - x \frac{d}{dx} P_n - (n+1) \, P_n = 0$$

with $P_n(1) = 1$ and $P_0(x) = 1$. Using the above recursion relations and the two conditions given, find the generating function $g(t,x)$ of the Legendre polynomials.

C2: **(a)** Assume that there exists an orthonormal basis $|e_i\rangle$, $(i = 1, 2, \cdots)$ in $L^2_w(a,b)$. Then, for any $|f\rangle \in L^2_w(a,b)$, the sequence of vectors $|f_k\rangle = \sum_{i=1}^{k} f^i \, |e_i\rangle$ with $f^i = \langle e_i \,|\, f\rangle$, $i = 1, 2, \cdots$ has $|f\rangle$ as the limit vector in the sense that

$$\lim_{k \to \infty} \rho(|f\rangle, |f_k\rangle) = 0.$$

(b) Prove that any orthogonal family $|e_i\rangle$, $i = 1, 2, \cdots$ in $L^2_w(a,b)$ is linearly independent.

(c) If $(e_1, e_2, \cdots, e_n)$ is a finite orthonormal family of functions in $L^2_w(a,b)$ and $f \in L^2_w(a,b)$, then

$$\| \, f - \sum_{k=1}^{n} f^k \, | \, e_k \rangle \, \|$$

has its minimum value when $f^i = \langle f \,|\, e_i \rangle$, $i = 1, 2, \cdots$ and

$$\sum_{k=1}^{n} |\langle f \,|\, e_k \rangle|^2 \leq \| \, f \, \|^2 \ .$$

C3: Using the Green's function technique solve the following Dirichlet problem.

$$\nabla^2 \, u = 0, \quad -\infty < x, y < \infty, \quad z > 0,$$
$$u(x,y,0) = f(x,y), \quad -\infty < x, y < \infty.$$

where $f(x,y)$ is a continuous function in $\mathbb{R}^2$.

C4: Determine the natural boundary condition at $x = b$ for the variational problem defined by

$$J(y) = \int_a^b L(x, y, y') \, dx + G(y(b)), \quad y \in \mathbb{C}^2[a,b], \quad y(a) = y_0$$

where G is a given differentiable function. As an application of this problem let $L = (y')^2, y \in [0,1], y(0) = 1$, $y(1)$ is unspecified and $G = y^2$. Find y extremizing this problem. Classify the extreme points. (Give reasons for your answers.)

Section D: Geometry and Topology

D1: State Bézout's Theorem. Show that if X and C are curves in $\mathbb{P}^2$, then the degree of the divisor $D = X \cap C$ is $\deg(X)\deg(C)$.

D2: Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $\mathrm{U}(M) \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$\mathrm{U}(M) = \{(p, v) \in \mathrm{T}(\mathbb{R}^n) | \, p \in M, v \in \mathrm{T}_p(M), \|v\| = 1\}$$

It is called the unit tangent bundle of M. Prove that $\mathrm{U}(M)$ is an embedded $(2m - 1)$-dimensional submanifold of $\mathrm{T}(\mathbb{R}^n) \approx \mathbb{R}^n \times \mathbb{R}^n$.

D3: State the Seifert–van Kampen Theorem. Prove that if U and V are path-connected open subsets of X such that $X = U \cup V$ and $U \cap V$ is non-empty and simply connected, then $\pi(X)$ is the free product of the groups $\pi(U)$ and $\pi(V)$ with respect to the homomorphisms $\varphi_1 \colon \pi(U) \to \pi(X)$ and $\varphi_2 \colon \pi(V) \to \pi(X)$.

D4: Let $\varphi \colon X \to Y$ be a closed (it maps closed sets to closed sets) continuous surjective (onto) map such that $\varphi^{-1}(y)$ is compact for each $y \in Y$. Show that:

(a) if X is Hausdorff then so is Y.

(b) if X is regular then so is Y.

(c) if X is locally compact then so is Y.

(d) if X is second countable then so is Y.