BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

3 September 2013

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3.

Examiner 2: Analysis, questions B1, B2, B3, B4.

Examiner 3: Applied Mathematics, questions C1, C2, C3, C4.

Examiner 4: Geometry, questions D1, D2.

Examiner 5: Topology, questions D3, D4.

Time allowed: three hours.

Section A: Algebra

A1: Let R be a ring, M an R-module, N an R-submodule of M.

(a) Show that there exists an R-submodule L of M that is maximal subject to the condition that $N \cap L = \{0\}$.

(b) Give an example where infinitely many such L exist.

(c) Give an example where N is a proper submodule of M and $L = \{0\}$.

A2: Let $f_1(X)$, ..., $f_n(X)$ be polynomials over $\mathbb{Q}$ with degree 2. Let L be the splitting field for $f_1(X)...f_n(X)$ over $\mathbb{Q}$.

(a) Show that the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is abelian. (Hint: consider the squares of the group elements.)

(b) Now suppose that the degree of L over $\mathbb{Q}$ is $|L : \mathbb{Q}| = 8$. How many fields K are there such that $\mathbb{Q} \leq K \leq L$?

A3: **(a)** Let χ be an ordinary character of a finite group G. Suppose that $\chi(1) = 2$ and $\chi(g) \neq 2$ for all $g \in G - \{1\}$. Let $a \in G - Z(G)$ such that $a^3 = 1$. Show that $\chi(a) \in \{-1, -\omega, -\omega^2\}$ where $\omega = e^{2\pi i/3}$. (Hint: consider the eigenvalues of the action of a.)

(b) Now let G be such that $|G| = 24$ and G has the quaternion group Q_8 as a normal subgroup and G has an element a with order 3 such that, writing $Q_8 = \{1, z, i, iz, j, jz, k, kz\}$ with $z^2 = 1$, then $aia^{-1} = j$ and $aja^{-1} = k$. You may assume that the conjugacy classes are

$$[1]\,,\ \ [z]\,,\ \ [i]\,,\ \ [a]\,,\ \ [a^2]\,,\ \ [az]\,,\ \ [a^2z]$$

and that the sizes of the conjugacy classes are, respectively,

$$1\,,\quad 1\,,\quad 6\,,\quad 4\,,\quad 4\,,\quad 4\,,\quad 4\,.$$

Explaining your methods, find the character table of G.

Section B: Analysis

B1: Let K be a compact subset of $\mathbb{R}$. Suppose that there exists a bounded (in the uniform norm) sequence of functions on K which is not equicontinuous. Prove that for any measure μ with $\operatorname{supp}\mu = K$ and for each polynomial P we have that $\int P^2\,d\mu = 0$ implies $P \equiv 0$. (Hint: prove that the set K is not finite.)

B2: Let $(\mu_\alpha)_{\alpha\in A}$ be a family of regular signed Borel measures on a compact set K. Suppose that for each continuous on K function f we have

$$\sup_{\alpha\in A} \Big| \int f\,d\mu_\alpha \Big| < \infty.$$

Show that for each constant M we have

$$\sup_{f\in B_M} \sup_{\alpha\in A} \Big| \int f\,d\mu_\alpha \Big| < \infty$$

where $B_M = \{f \in C(K) : |f(x)| \leq M \text{ for } x \in K\}$.

B3: Define $f(z) = \displaystyle\int_0^1 \frac{dt}{t-z}$ for $z = x + iy \in \mathbb{C} \setminus [0,1]$. In each of the following three cases, compute the limit $\displaystyle\lim_{y\to 0^+} \big[f(x+iy) - f(x-iy)\big]$.

(a) $0 < x < 1$,

(b) $x = 0$ or $x = 1$,

(c) $x > 1$ or $x < 0$.

B4: For $n = 1, 2, \ldots$, let $\{a_n\}$ be a sequence of distinct complex numbers with $a_n \to \infty$ as $n \to \infty$ and let $\{c_n\}$ be any sequence of complex numbers. Prove that there exists an entire function f such that $f(a_n) = c_n$ for every n.

Section C: Applied Mathematics

C1: Let a sequence of functions be given by

$$D_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2}, \quad n = 1, 2, \cdots, \quad x \in \mathbb{R}$$

For any good function $f(x)$, calculate the following:

(a) $\lim_{n\to\infty} D_n(x)$,

(b) $\int_{-\infty}^{\infty} D_n(x)\, dx$,

(c) $\lim_{n\to\infty} \int_{-\infty}^{\infty} D_n(x)\, f(x)\, dx$.

C2: Let $P_n(x)$, $(n = 0, 1, 2, \cdots)$ be the Legendre polynomials defined through the Rodriguez formula

$$P_n(x) = \frac{(-1)^n}{2^n\, n!}\, \frac{d^n}{dx^n}\, (1 - x^2)^n\,.$$

[Further information: $w = 1, s = 1 - x^2$, and $k_n = \dfrac{2^n\, \Gamma(n + 1/2)}{n!\, \Gamma(1/2)}$, $k'_n = 0$, $h_n = \dfrac{1}{n + 1/2}$.]

(a) Show that

$$\int_{-1}^{1} P_n(x)\, h_m(x)\, dx = 0, \quad m < n$$

where $h_m(x)$ is a polynomial of degree m. Discuss the case when $m \geq n$.

(b) Prove that $P_n(x)$ is a polynomial of degree n.

(c) Prove the recursion relation $(n + 1)P_{n+1} - (2n + 1)x\, P_n + nP_{n-1} = 0$.

C3: Obtain the Euler-Lagrange equation and the associated natural boundary conditions for the problem $\delta J = 0$ where

$$J(y) = \int_a^b L(x, y, y')dx - \beta y(b) + \alpha y(a)$$

Here α and β are arbitrary constants and $y(a)$ and $y(b)$ are not prescribed.

C4: Show that the following Dirichlet problem in a rectangular region $D \subset \mathbb{R}^2$ is well-posed:

$$\begin{aligned} \nabla^2 u &= 0, \quad (x, y) \in D \quad [\, 0 < x < a, \quad 0 < y < b\,], \\ u(x, 0) &= f(x), \quad u(x, b) = 0, \quad 0 \leq x \leq a, \\ u(0, y) &= u(a, y) = 0, \quad 0 \leq y \leq b. \end{aligned}$$

Section D: Geometry and Topology

D1: Let C be a compact Riemann surface of genus g. For any divisor D on C, the Riemann-Roch theorem says that

$$\ell(D) - \ell(K - D) = \deg D - g + 1,$$

where K is a canonical divisor of C. If further $D \geq 0$, $\ell(D) > 0$ and $\ell(K - D) > 0$, we say D is a special divisor. The Clifford theorem says that for any special divisor D on C,

$$\ell(D) \leq \frac{1}{2} \deg D + 1.$$

Prove the Clifford theorem assuming the Riemann-Roch theorem.

D2: Let C be a compact Riemann surface of genus 1. Let K be a canonical divisor on C. We know that $\ell(K) = 1$ from various considerations. Fix a point $p \in C$. Use the Riemann-Roch theorem to prove the following.

(a) There exist nontrivial meromorphic functions f and g on C with the properties that f and g are holomorphic everywhere on C except at the point p where f and g have poles of orders 2 and 3 respectively.

(b) There exists a polynomial

$$P(X, Y) = c_0 + c_1X + c_2Y + c_3X^2 + c_4XY + c_5X^3 + c_6Y^2,$$

where c_i are complex numbers with $c_5c_6 \neq 0$ such that $P(f, g) \equiv 0$.

D3: Write I for unit interval $[0, 1]$, write $\mathbf{D}^2$ for the unit disk $\{(x, y) \in \mathbf{R}^2 \,|\, x^2 + y^2 \leq 1\}$, write $\mathbf{S}^2$ is the unit sphere $\{(x, y, z) \in \mathbf{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ and $\mathbf{R}P^2$ for the real projective space $\mathbf{S}^2/\sim$ where $(x, y, z) \sim (-x, -y, -z)$ for (x, y, z) in $\mathbf{S}^2$. Take any point $x_0 \in \mathbf{R}P^2$. Let $A = \{0, 1\}$ and $B = \{(-1, 0), (1, 0)\}$. Let X be the space of continuous functions $(I, A) \to (\mathbf{R}P^2, \{x_0\})$ and let Y be the space of continuous functions $(\mathbf{D}^2, B) \to (\mathbf{R}P^2, \{x_0\})$, both spaces having the compact-open topology.

(a) Is X homotopy equivalent to Y ?

(b) Find the cardinality of the path connected components of X.

D4: Let $\mathbf{S}^n$ denote the unit sphere in the Euclidean space $\mathbf{R}^{n+1}$. Let T denote the torus $\mathbf{S}^1 \times \mathbf{S}^1$.

(a) Calculate $H_n(T; \mathbf{Z})$, the n^{th} singular homology group of T, for $n \geq 0$.

(b) Calculate $H^n(T; \mathbf{Z}/3)$, the n^{th} singular cohomology group of T with coefficients in $\mathbf{Z}/3$, for $n \geq 0$.

(c) Is there a continuous function from $\mathbf{S^2}$ to T which induces an isomorphism from $H_2(\mathbf{S^2}; \mathbf{Z})$ to $H_2(T; \mathbf{Z})$?

(d) Is there a continuous function from T to $\mathbf{S^2}$ which induces an isomorphism from $H_2(T; \mathbf{Z})$ to $H_2(\mathbf{S^2}; \mathbf{Z})$?

Comments and solutions

A1: (Sketch.) (a) Zorn's Lemma. (b) Complementary spaces of subspaces of vector spaces over infinite fields. (c) Proper subgroups of cyclic groups, as $\mathbb{Z}$-modules.

A2: The Galois group acts faithfully on the roots and all the orbits have size at most 2, so the group is an elementary abelian 2-group. When the group has order 8, the number of subgroups with order 1, 2, 4, 8 is 1, 7, 7, 1, respectively. By the Fundamental Theorem of Galois Theory, the number of intermediate fields is $1 + 7 + 7 + 1 = 16$.

A3: (Sketch.) (a) The representation is faithful, so the eigenvalues of a must be distinct cube roots of unity. (b) The irreducibles of degree unity can be obtained by inflation from $G/Q_8 \cong C_3$. An irreducible with degree 3 can be obtained by inflation from $G/\langle z \rangle \cong A_4$ or by induction from Q_8. The three irreducibles ψ with degree 2 all have $\psi(z) = -2$ by column orthonormality. Bearing in mind part (a) and tensor products with degree unity irreducibles, the rest of the table is forced. Note that $\psi(az) = -\psi(a)$ because z acts as multiplication by -1. If a few of these tricks are missed, column orthonormality may make up for it. Incidentally, $G \cong \mathrm{SL}(2, 3)$.

D3: Call the limit L. By Morera theorem, f is holomorphic and hence continuous wherever it is defined. So if $x > 1$ or $x < 0$, then $L = 0$. Next let $0 \le x \le 1$ and $y > 0$. We have

$$\begin{aligned} f(x+iy) &= \int_0^1 \frac{dt}{t-x-iy} = \int_0^1 \frac{t-x+iy}{(t-x)^2+y^2}\,dt \\ &= \int_0^1 \frac{t-x}{(t-x)^2+y^2}\,dt + \frac{i}{y}\int_0^1 \frac{dt}{1+\left(\frac{t-x}{y}\right)^2} \\ &= \left[\frac{1}{2}\ln\bigl((t-x)^2+y^2\bigr) + i\tan^{-1}\Bigl(\frac{t-x}{y}\Bigr)\right]_{t=0}^{t=1} \\ &= \frac{1}{2}\ln\frac{(1-x)^2+y^2}{x^2+y^2} + i\left[\tan^{-1}\Bigl(\frac{1-x}{y}\Bigr) + \tan^{-1}\Bigl(\frac{x}{y}\Bigr)\right], \end{aligned}$$

where all evaluations are real. Then

$$f(x+iy) - f(x-iy) = 2i\left[\tan^{-1}\Bigl(\frac{1-x}{y}\Bigr) + \tan^{-1}\Bigl(\frac{x}{y}\Bigr)\right].$$

Thus $L = 2\pi i$ if $0 < x < 1$, and $L = \pi i$ if $x = 0$ or $x = 1$.

D4: By the Weierstrass theorem, there is an entire function g with a simple (since the a_n are distinct) zero at each a_n and no other zeros. So $g(a_n) = 0$ and $g'(a_n) = b_n \neq 0$ for each n; that is, g has the Taylor expansion $g(z) = b_n(z-a_n) + \dfrac{g''(a_n)}{2}(z-a_n)^2 + \cdots$ at each a_n. By the Mittag-Leffler theorem, there is a meromorphic function h in $\mathbb{C}$ with the principal part $\dfrac{c_n}{b_n(z-a_n)}$ at each a_n and no other poles; that is, h has the Laurent expansion $h(z) = \dfrac{c_n}{b_n(z-a_n)} + h_n(z)$ at each a_n, where h_n is holomorphic in a neighborhood of a_n. Set $f = gh$. Then f is entire and satisfies $f(a_n) = c_n$ for every n.