BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

9 July 2012

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Invariant Theory, question A1.

Examiner 2: Algebra, questions A2, A3, A4.

Examiner 3: Real Analysis, questions B1, B2.

Examiner 4: Complex Analysis, questions B3, B4.

Examiner 5: Applied Mathematics, questions C1, C2, C3, C4.

Examiner 6: Topology, questions D1, D2.

Time allowed: three hours.

Section A: Algebra

A1: Let S denote the polynomial ring $F[x, y, z]$ over a field F. Let $f_1 = x^2$, $f_2 = y^2$ and $f_3 = z^2$. Write down the Kozsul complex associated with the sequence f_1, f_2, f_3 of elements in S. Write down all the differentials in matrix form. Compute all homologies in the complex.

A2: Let p, q, r be distinct primes. Show that $[\mathbb{Q}[\sqrt{p}] : \mathbb{Q}] = 2$ and $[\mathbb{Q}[\sqrt{p}, \sqrt{q}] : \mathbb{Q}] = 4$ and $[\mathbb{Q}[\sqrt{p}, \sqrt{q}, \sqrt{r}] : \mathbb{Q}] = 8$. Find the Galois group $\mathrm{Gal}(\mathbb{Q}[\sqrt{p}, \sqrt{q}, \sqrt{r}]/\mathbb{Q})$. How many fields K are there such that $\mathbb{Q} \leq K \leq \mathbb{Q}[\sqrt{p}, \sqrt{q}, \sqrt{r}]$?

A3: **(a)** Let A be a semisimple algebra over $\mathbb{C}$ with exactly n isomorphism classes of simple modules. Using the Artin–Wedderburn Structure Theorem, show that A has exactly 2^n two-sided ideals.

(b) Now suppose that A is 7-dimensional and non-commutative. Up to isomorphism, how many quotient rings does A have, including the "zero ring" with only one element? (Hint: consider the quotients by the two-sided ideals, then collect the quotients into isomorphism classes.)

A4: **(a)** Consider a finite group G and a finite G-set Ω. Let χ_Ω denote the character of the permutation $\mathbb{C}G$-module $\mathbb{C}\Omega$. Write down a formula for the character value $\chi_\Omega(g)$, where $g \in G$. Briefly justify your formula.

(b) Now let $G = A_5$ and let Ω be the natural permutation set (with size 5). In this case, find the character values and show that $\chi_\Omega - \chi_0$ is irreducible, where χ_0 denotes the trivial character.

(c) Repeat part (c) in the case where Ω is the set of Sylow 5-subgroups of G under the conjugation action.

(d) Using parts (b) and (c) together with the row and column orthogonality relations, find the character table of A_5. (No marks will be awarded for arguments using other methods, such as induction from subgroups or assumption of the existence of the dodecahedron and icosahedron.)

Section B: Analysis

B1: Given a Borel measure μ on $\mathbb{R}$, by $supp\,\mu$ we denote a closed set S such that $\mu(\mathbb{R} \setminus S) = 0$ and for every open set V with $V \cap S \neq \emptyset$ we have $\mu(V) > 0$. Show that $supp\,\mu$ contains infinitely many points if and only if $\int P^2\, d\mu = 0$ implies $P \equiv 0$ for any polynomial P.

B2: Show that the space $(l_1\,, \sigma(l_1, c_0))$, that is the space l_1 with its weak* topology, is of the first category in itself.

B3: Let $\mathbb{D} = \{\, z \in \mathbb{C} : |z| < 1 \,\}$ be the unit disc, $c \in \mathbb{D}$ and $V = \mathbb{D} \setminus \{c\}$.

(a) Prove that a holomorphic map $f : V \to V$ extends to a holomorphic map $F : \mathbb{D} \to \mathbb{D}$.

(b) Prove that if f is one-to-one and onto, then F is also one-to-one and onto.

(c) Find all holomorphic automorphisms of V.

B4: Let $f = p/q$ be a rational function, where p and q are polynomials of degree m and n, respectively, that are relatively prime, i.e., with no common factors. The *degree* d of f is defined to be the larger of m and n. Let $w \in \mathbb{C}$ so that $w \neq \lim\limits_{z \to \infty} f(z)$. Use the argument principle to prove that f takes the value w exactly d times counted with multiplicity. (It might help to treat the cases $m > n$, $m = n$, and $m < n$ separately.)

Section C: Applied Mathematics

C1: Prove that the classical orthogonal polynomial $C_n(x)$ with $x \in I = [a, b]$ has n distinct zeros in I. (Recall that $C_n(x) = \dfrac{1}{w}\dfrac{d^n}{dx^n}(ws^n), n = 0, 1, 2, \cdots$ where $w(x)$ is positive function in $[a, b]$, $s(x)$ is a polynomial of degree less or equal to two and $C_1(x)$ is polynomial of degree one. The functions w and s satisfy the boundary conditions $w(a)s(a) = w(b)s(b) = 0$.)

C2: Consider $u'' + u = f(x)$ with $u(0) = 0$ and $u(a) = 0$ with $\sin a \neq 0$.

(a) Show that

$$u(x) = \int_0^x f(y)\sin(x-y)dy + c\sin x$$

where c is such that $c\sin a = -\int_0^a f(y)\sin(a-y)dy$. Hence

$$u(x) = \frac{1}{\sin a}\left[\int_0^x f(y)\sin(x-y)\,\sin a dy + \int_0^a f(y)\sin(a-y)\sin x\,dy\right].$$

(b) Prove that, when $\sin a \neq 0$, we have $G(x,y) = \begin{cases} \dfrac{\sin y\,\sin(x-a)}{\sin a} & \text{if } y \leq x, \\ \dfrac{\sin(y-a)\,\sin x}{\sin a} & \text{if } x \leq y. \end{cases}$

(c) Show that, if $\sin a = 0$, then the above equations have no solution unless $f(x)$ satisfies the condition

$$\int_0^a f(x)\sin(a-x)dx = 0$$

in which case there are infinitely many solutions , each of the form

$$u(x) = \int_0^x f(y)\,\sin(x-y)dy + C\sin x$$

where C is an arbitrary constant.

C3: Consider the integral equation $y(x) = f(x) + \lambda\int_a^b e^{x-s}\,y(s)ds$.

(i) Find the solution.

(ii) Find the cases (restrictions on λ) where there is no solution.

(iii) Find the cases where the solution is not unique.

(iv) Find the solution when $f(x) = x^2$.

C4: Let $u \in C^2(D) \cap C^1(\bar{D})$, where D is a bounded domain and B is its boundary. Show that:

(i) If $\nabla^2 u \geq 0$ in D, then $u(x) \leq M = max_{y\in B}\, u(y)$ for all $x \in D$.

(ii) If $\nabla^2 u \leq 0$ in D, then $u(x) \geq m = \min_{y\in B}\, u(y)$ for all $x \in D$.

(iii) If $\nabla^2 u = 0$ in D, then $m \leq u(x) \leq M$ for all $x \in D$.

Section D: Topology

D1: Let $p : \widetilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite for all $x \in X$. Show that $\widetilde{X}$ is compact Hausdorff if and only if X is compact Hausdorff.

D2: Let X denote the space obtained from S^n by attaching a cell e^{n+1} by a map of degree m. Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\widetilde{H}_i(-, \mathbb{Z})$ for all i, but not on $H^{n+1}(-, \mathbb{Z})$. Conclude that the splitting in the universal coefficient theorem for cohomology cannot be natural.