BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

20 September 2011

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3, A4.

Examiner 2: Real Analysis, questions B1, B2.

Examiner 3: Complex Analysis, questions B3, B4.

Examiner 4: Applied Mathematics, questions C1, C2, C3, C4.

Examiner 5: Topology, questions D1, D2.

Examiner 6: Algebraic Number Theory, question D3.

Examiner 7: Analytic Number Theory, question D4.

Time allowed: three hours.

Section A: Algebra

A1: Let G_n be the group of symmetries of an n-dimensional cube, in other words, the group of isometries $\mathbb{R}^n \to \mathbb{R}^n$ that permute the points $(\pm 1, \pm 1, ..., \pm 1)$. For which values of n is G_n solvable?

A2: Give an example of a non-trivial group which does not have a maximal subgroup. Let G be a non-trivial group and let $1 \neq g \in G$. Show that G has a subgroup H that is maximal subject to the condition that $g \notin H$.

A3: Let G be the group of order 55 with generators a and b such that $a^{11} = b^5 = 1$ and $bab^{-1} = a^4$.

(a) Briefly, show that the elements 1, a, a^{-1}, b, b^2, b^3, b^4 comprise a set of representatives of the conjugacy classes of G, the orders of the conjugacy classes being 1, 5, 5, 11, 11, 11, 11, respectively.

(b) Find the complex character table of G.

(c) Let ϕ be a non-trivial irreducible complex character of the Sylow 11-subgroup of G. Let χ be the complex character of G induced from ϕ. Show that χ is irreducible.

A4: Let $0 \to M_0 \to ... \to M_n \to 0$ be an exact sequence of modules of a unital ring. Starting from the definition of an Artinian module, show that if n of the modules M_0, ..., M_n are Artinian, then all $n+1$ of them are Artinian. Does a similar conclusion hold for Noetherian modules? For semisimple modules?

Section B: Analysis

B1: Let (X, μ) be a measure space, $f \in L(\mu)$. Show that the set $\{x : f(x) \neq 0\}$ is $\sigma-$ finite.

B2: Suppose $(x_n)_{n=1}^{\infty} \in l_2$, $(y_n)_{n=1}^{\infty} \in l_3$. Prove that $\left(\frac{x_n\, y_n}{\sqrt[5]{n}}\right)_{n=1}^{\infty} \in l_1$.

B3: **(a)** Suppose that $c_n \in \mathbb{C}$ and that $\sum_{n=0}^{\infty} c_n$ converges to A. Define $f(r) = \sum_{n=0}^{\infty} c_n\, r^n$ for $0 \leq r < 1$. Prove that $\lim_{r \to 1^-} f(r) = A$. (Set $s_n = c_0 + \cdots + c_n$ and write $f(r)$ in terms of s_n.)

(b) Find $\{c_n\}$ such that $\sum_{n=0}^{\infty} c_n$ does not converge, but $\lim_{r \to 1^-} f(r)$ exists.

B4: Suppose $a_n, b_n \in \mathbb{C}$ with $\sum_{n=0}^{\infty} |a_n - b_n| < \infty$. Determine the largest set $D \subset \mathbb{C}$ on which $g(z) = \prod_{n=1}^{\infty} \frac{z - a_n}{z - b_n}$ defines a holomorphic function.

Section C: Applied Mathematics

C1: Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \leq \frac{-1}{n} \\ \frac{(nx+1)}{2} & \text{if } \frac{-1}{n} \leq x \leq \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases}$$

where $n = 1, 2, 3, \cdots$

(a) (In the sense of generalized functions.) Prove that $h_n(x) \to \theta(x)$ where $\theta(x)$ is the step function and

(b) $\dfrac{dh_n(x)}{dx} \to \delta(x)$. Hence formally we may write that $\dfrac{d\theta(x)}{dx} = \delta(x)$.

C2: Find the series solution the differential equation $2u'' + zu' + 3u = 0$ about the point $z = 0$.

C3: If ℓ is not preassigned, show that the stationary functions corresponding to the problem $\delta \int_0^\ell [y'^2 + 4(y - \ell)]dx = 0$, with $y(0) = 2$ and $y(\ell) = \ell^2$, are of the form $y = x^2 - 2(x/\ell) + 2$, where ℓ is one of the roots of the equation $2\ell^4 - 2\ell^3 - 1 = 0$.

C4: Find the solution of the heat equation $u_t = u_{xx} + xe^{-t}$ with

$$u(0,t) = 0\ , \qquad u_x(1,t) + u(1,t) = 0\ , \qquad u(x,0) = 0\ .$$

Section D: Topology and Number Theory

D1: Let f and g denote the homotopic maps from S^{n-1} to Y where Y is a Hausdorff space and $n \geq 1$. Let D^n denote the n-ball with $\partial D^n = S^{n-1}$. Prove that $D^n \cup_f Y$ and $D^n \cup_g Y$ have the same homology groups.

D2: A triangulation of a topological space X is a homeomorphism from the geometric realization of a simplicial complex to X. Find a triangulation of $\mathbf{R}P^2$ and compute its simplicial homology.

D3: **(a)** Prove that $f(x) = x^3 + 2x^2 + 4$ is irreducible in $\mathbb{Q}(x)$.

(b) Let α be a root of $f(x)$ and assume that the irreducible polynomial of α^2 is $g(x) = x^3 - 4x^2 - 16x - 16$. Prove that $disc(\alpha) = -16.5.7$

(c) Let $R = \mathbb{A} \cap \mathbb{Q}[\alpha]$. Prove that $\alpha^2/2 \in R$ but $\alpha^2/4 \notin R$.

(d) Prove that if $(a + b\alpha)/2 \in R$ with a and b are integers, then both a and b are even.

(e) Prove that 1, α, $\alpha^2/2$ is an integral basis of R.

(f) Find $\operatorname{disc}(R)$.

D4: Prove Turán's formula, i.e., show that for $x \geq 3$,

$$\sum_{n \leq x} (\omega(n) - \log\log x)^2 \ll x \log\log x$$

where $\omega(n) = \sum_{p|n} 1$ is the number of distinct prime divisors of n.

Some solutions

D3: (a) Convergence of $\sum_{n=0}^{\infty} c_n$ implies that $\{c_n\}$ is bounded. Then the sum defining $f(r)$ converges (absolutely) for $0 \le r < 1$. Define s_n as suggested; also set $s_{-1} = 0$. Then

$$\sum_{n=0}^{m} c_n\, r^n = \sum_{n=0}^{m} (s_n - s_{n-1})\, r^n = (1-r) \sum_{n=0}^{m-1} s_n\, r^n + s_m r^m$$

for $0 \le r < 1$. Letting $m \to \infty$, we obtain $f(r) = (1-r) \sum_{n=0}^{\infty} s_n\, r^n$.

Let $\varepsilon > 0$. Fix N such that for $n > N$, we have $|s_n - A| < \varepsilon$. Then

$$\begin{aligned} |f(r) - A| &= \left| (1-r) \sum_{n=0}^{\infty} s_n\, r^n - A\,(1-r) \sum_{n=0}^{\infty} r^n \right| = \left| (1-r) \sum_{n=0}^{\infty} (s_n - A)\, r^n \right| \\ &\le (1-r) \sum_{n=0}^{N} |s_n - A|\, r^n + (1-r)\,\varepsilon \sum_{n=N+1}^{\infty} r^n \le (1-r)\, B + \varepsilon, \end{aligned}$$

where B is finite. Letting $r \to 1^-$ and noting that $\varepsilon > 0$ is arbitrary, we are done.

(b) Consider $c_n = (-1)^n$; obviously $\sum_{n=0}^{\infty} c_n$ does not converge. But

$$\lim_{r \to 1^-} f(r) = \lim_{r \to 1^-} \sum_{n=0}^{\infty} (-1)^n\, r^n = \lim_{r \to 1^-} \frac{1}{1+r} = \frac{1}{2}\,.$$

D4: First note that $\prod_{n=1}^{\infty} \frac{z - a_n}{z - b_n} = \prod_{n=1}^{\infty} \left(1 + \frac{b_n - a_n}{z - b_n} \right)$. Set $D = \mathbb{C} \setminus \overline{\{b_1, b_2, \ldots\}}$, where the overbar means closure. Let K be a compact subset of D. There is a $d > 0$ such that $|z - b_n| \ge d$ for all $z \in K$ and for all $n = 1, 2, \ldots$. Then for all $z \in K$,

$$\sum_{n=1}^{\infty} \left| \frac{b_n - a_n}{z - b_n} \right| \le \frac{1}{d} \sum_{n=1}^{\infty} |b_n - a_n| < \infty;$$

that is, this sum converges uniformly on K by the Weierstrass M-test. Consequently, the infinite product converges uniformly on K. Then g is holomorphic on D since the factors of the infinite product defining it are holomorphic on D.