BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

15 June 2010

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3, A4.

Examiner 2: Real Analysis, questions B1, B2.

Examiner 3: Complex Analysis, questions B3, B4.

Examiner 4: Applied Mathematics, questions C1, C2, C3, C4.

Examiner 5: Analytic number theory, question D1.

Examiner 6: Algebraic number theory, question D2

Time allowed: three hours.

Section A: Algebra

A1: Let F be a field with characteristic zero, let $f(X)$ be an irreducible quartic (degree 4) polynomial over F, let E be the splitting field for $f(X)$ over F, and let G be the Galois group of the extension E/F.

(a) State the Fundamental Theorem of Galois Theory, including clauses concerning the order of the Galois group and the normal subgroups of the Galois group.

(b) Let α_1, α_2, α_3, α_4 be the roots to $f(X)$, and let

$$\delta = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4)(\alpha_3 - \alpha_4) \, .$$

Suppose that $\delta \in F$. Show that $G \cong V_4$ or $G \cong A_4$.

(c) In each of the two cases in part (b), find the number of intermediate fields $F \leq L \leq E$. In each of those two cases, how many of those L are normal extensions of F?

A2: Let G be a finite group and let p be the smallest prime divisor of $|G|$.

(a) Let H be a subgroup of G with index $|G : H| = p$. Show that H is normal in G. (Hint: consider a suitable group homomorphism from G to the symmetric group S_p.)

(b) Let K be the intersection of all the subgroups G satisfying the property in (a). Show that G/K is an elementary abelian p-group (we mean, G/K is isomorphic to a direct product of cyclic groups with order p).

A3: Let F be a field. Up to isomorphism of algebras over F, how many 3-dimensional semisimple algebras over F are there in the following three cases: $F = \mathbb{C}$, $F = \mathbb{R}$, $F = \mathbb{Q}$? In those cases where there are infinitely many isomorphism classes, are there countably many or uncountably many?

A4: Let G be the group with order 27 generated by the elements a, b, c with the following relations:

$$a^3 = b^3 = c^3 = 1 \, , \qquad ac = ca \, , \qquad bc = cb \, , \qquad ba = abc \, .$$

Construct the ordinary character table of G, justifying your steps. (Hint: first note that any element of G can be written uniquely in the form $a^\alpha b^\beta c^\gamma$ where $\alpha, \beta, \gamma \in \{0, 1, 2\}$.)

Section B: Analysis

B1: Given $f \in L(\mathbb{R})$ and $h > 0$, let $g_h(x) = \dfrac{1}{2h} \displaystyle\int_{x-h}^{x+h} f(t)\, dt$. Prove that $g_h \in L(\mathbb{R})$ and $\displaystyle\int_{\mathbb{R}} |g_h(x)|\, dx \leq ||\, f\,||_1$.

B2: Given sequence $a = (a_n)$, consider the diagonal operator $T_a : l_3 \longrightarrow l_2 : (x_n) \mapsto (a_n\, x_n)$. Find a criterion of boundedness of T_a in terms of the sequence a. In the case of bounded T_a, find its norm.

B3: Let $\alpha \in \mathbb{C}$ be such that $|\alpha| \neq 1$. Compute $I(\alpha) = \displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha \cos\theta + \alpha^2}$ using complex integration.

B4: Let γ be smooth curve in $\mathbb{C}$, and set $D = \mathbb{C} \setminus \gamma$. Suppose f is a continuous function on γ, and define $g(z) = \displaystyle\int_\gamma \frac{f(w)}{w - z}\, dw$ for $z \in D$. Prove that g is holomorphic in D and $g'(z) = \displaystyle\int_\gamma \frac{f(w)}{(w - z)^2}\, dw$ for $z \in D$.

Section C: Applied Mathematics

C1: Let $u'' - w^2\, u = f(x), \quad x \in (a, b)$. Here w is a real number and $aw \neq n\pi$ and $bw \neq n\pi$ where $n = 0, 1, 2, \cdots$. Solve this problem by using the method of Green's Function with the boundary conditions to $u(a) = 0$ and $u(b) = 0$.

C4: Show that the large eigenvalues of the problem

$$y'' + \lambda(x + \pi)^4\, y = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

are given by

$$\lambda = \lambda_n = \frac{9n^2}{49\pi^4}$$

for large integers n , and find the corresponding eigenfunctions

C3: Find the solution of the heat equation $u_t = u_{xx} + \sin x\, e^{-t}, \quad x \in (0, \pi),\ t > 0$ with $u(0, t) = 0, \quad u(\pi, t) = 0$ and $u(x, 0) = u_0 \sin 2x$, where u_0 is the initial temperature.

C4: **(a)** Obtain the most general solution of the integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x - s)\, y(s) ds.$$

Discuss all possible cases.

(b) Obtain solutions when when $f(x) = x$ and $f(x) = 1$.

Section D: Number Theory

D1: (a) Show that

$$\sum_{p_1 \neq p_2} \left\lfloor \frac{x}{p_1 p_2} \right\rfloor = x \sum_{p_1 p_2 \leq x} \frac{1}{p_1 p_2} + O(x).$$

(b) Explain why the sum on the right above is

$$\left(\sum_{p \leq x} \frac{1}{p} \right)^2 - 2 \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} \sum_{x/p_1 < p_2 \leq x} \frac{1}{p_2} + \left(\sum_{\sqrt{x} < p \leq x} \frac{1}{p} \right)^2. \tag{1}$$

(c) Show that if $y \leq \sqrt{x}$, then

$$\sum_{x/y < p \leq x} \frac{1}{p} = \log\log x - \log\log(x/y) + O(1/\log x).$$

(d) Show that the right-hand side above is $\asymp (\log y)/\log x$.

(e) Deduce that the second and third terms in (1) are $\ll 1$.

D2: (a) Find the ideal class group of $\mathbb{A} \cap Q(\sqrt{-35})$.

(b) Find the ideal class group of $\mathbb{A} \cap Q(\sqrt{-42})$ and show that if p is a prime with $\left(\frac{-168}{p}\right) = 1$ then p is represented by exactly one the following forms $x^2 + 42y^2$, $3x^2 + 14y^2$, $2x^2 + 21y^2$, $6x^2 + 7y^2$. Determine those primes p with $\left(\frac{-168}{p}\right) = 1$ that are represented by $6x^2 + 7y^2$. (Hint: look at if p is a square mod 3 or 7).