BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

5 January 2010

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3, A4.

Examiner 2: Real Analysis, questions B1, B2.

Examiner 3: Complex Analysis, questions B3, B4.

Examiner 4: Applied Mathematics, questions C1, C2, C3, C4.

Examiner 5: General Topology, questions D1, D2.

Examiner 6: Algebraic topology, questions D3, D4.

Time allowed: three hours.

Section A: Algebra

A1: State the Fundamental Theorem of Galois Theory, including clauses concerning the order of the Galois group and the normal subgroups of the Galois group. Consider the Fermat prime $p = 65537 = 2^{16}+1$. Let $\zeta = e^{2\pi i/p}$. Find that number of fields K such that $\mathbb{Q} \leq K \leq \mathbb{Q}[\zeta]$. You may assume that, for each integer j in the range $1 \leq j \leq p-1$, there exists an automorphism θ_j of $\mathbb{Q}[\zeta]$ given by $\theta(\zeta) = \zeta^j$.

A2: State Sylow's Theorem. Show that, up to isomorphism, there are precisely 5 groups with order $4p$, where p is as in the previous question. You may assume that the unit group $\mathbb{Z}/p - \{0\}$ is cyclic, where $\mathbb{Z}/p$ denotes the ring of integers modulo p.

A3: State Zorn's Lemma. Let R be a ring such that every left R-module is a sum of simple modules. Show that every left R-module is a direct sum of simple modules. (You may not assume any standard results about semisimple rings!)

A4: Assume that there exists a finite group whose conjugacy classes can be enumerated such that they have sizes 1, 21, 56, 42, 24, 24, and the character table has the following form.

1	1	1	1	1	1
$*$	$*$	$*$	$*$	$*$	$*$
$*$	$*$	$*$	$*$	$*$	$*$
6	2	0	0	-1	-1
7	-1	1	-1	0	0
8	0	-1	0	1	1

Determine the entries of the two rows marked as $* * * * * *$. Be clear about what results you are using.

Section B: Analysis

B1: Measurable functions $f \in \mathfrak{N}(X, \mu), g \in \mathfrak{N}(Y, \nu)$ are *equimeasurable* if

$$\mu\{x \in X : |f(x)| > t\} = \nu\{y \in Y : |g(y)| > t\}$$

for any $t \geq 0$.

(a) Show that for each $f \in \mathfrak{N}(X, \mu)$ there is a nondecreasing Lebesgue measurable function $g : [0, +\infty) \longrightarrow [0, +\infty]$ equimeasurable with f.

(b) Show that $\int_X |f| d\mu = \int_Y |g| d\nu$ for any equimeasurable integrable functions f and g.

B2: Let $A : L_2(0,1) \longrightarrow L_1(0,1) : f \mapsto \displaystyle\int_0^x t \cdot f(t)\, dt$. Show that the operator A is well-defined and continuous. Find its norm.

B3: Let U be a bounded open subset of $\mathbb{C}$ and $f : U \to U$ be a holomorphic function. Suppose there exists a point $p \in U$ such that $f(p) = p$ and $f'(p) = 1$. Prove that f is linear. (Without loss of generality, $p = 0$, but explain this. Then consider $f_k = f \circ f \circ \cdots \circ f$, where f appears k times.)

B4: Let f be holomorphic on all of $\mathbb{C}$ and $f(z) = \sum\limits_{n=0}^{\infty} c_n (z - a)^n$ be the Taylor expansion of f about $a \in \mathbb{C}$. Suppose for each $a \in \mathbb{C}$, $c_n = 0$ for at least one n. Prove that f is a polynomial. (Use a countability argument.)

Section C: Applied Mathematics

C1: Use the method of Frobenius to obtain the general solution of the following differential equation, valid near $z = 0$:

$$zu'' - u' + 4z^3 u = 0$$

C2: Find the dominating term of the following integral

$$\int_1^2 \sqrt{3+t}\, e^{\frac{\lambda}{t+1}} dt \quad \text{as} \quad \lambda \to \infty$$

C3: Homogeneous wave equation with initial and boundary values. Let

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \quad (t > 0, 0 < x < L), \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x), \quad (0 \le x \le L), \\ u(0,t) &= u(L,t) = 0, \quad (t \ge 0) \end{aligned}$$

Prove the following theorem.

Theorem: *Let $f(x)$ have a continuous fourth derivative and $g(x)$ have a continuous third derivative for $0 \le x \le L$, and $f(0) = f(L) = 0$, $f''(0) = f''(L) = 0$, and $g(0) = g(L) = 0$. Then the initial boundary value problem above has a solution given by*

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)] \sin(\lambda_n x),$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx,$$

$$B_n = \frac{2}{\lambda_n cL} \int_0^L g(x) \sin(\frac{n\pi x}{L}) dx.$$

where $\lambda_n = \frac{n\pi}{L}$.

C4: Discuss the qualitative (stability) behavior of the system

$$\frac{dx}{dt} = x(1 - x - y), \quad \frac{dy}{dt} = \frac{1}{4} y(3 - 4y - 2x)\,.$$

Section D: Topology

D1: We say that a topological space X is locally path connected at x if for every open set V containing x there exists a path connected, open set U with $x \in U \subseteq V$. The space X is said to be locally path connected if it is locally path connected at x for all x in X. Show that a connected and locally path connected space is path connected. Is the converse true?

D2: Let $p : E \to B$ be a covering map, X be a connected and locally path connected topological space, and $f : X \to B$ be a continuous function. Then show that the map f lifts to a map $f' : X \to E$ (in other words there exists a continuous function $f' : X \to E$ such that $p \circ f' = f$) if and only if $f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(E, e_0)) \subseteq \pi_1(B, b_0)$ where $f(x_0) = p(e_0) = b_0 \in B$.

D3: Let S_n denote the n-dimensional sphere. Fix a basepoint $x_0 \in S_n$. Let X denote the space obtained from $S^n \times S^n$ by identifying the pairs of points $(x, x_0) \sim (x_0, x)$ for all $x \in S^n$. Calculate the integral cohomology ring $H^*(X, \mathbb{Z})$.

D4: Let $f : S^{2n} \to S^{2n}$ be a continuous map. Show that there exists an $x \in S^{2n}$ with $f(x) = x$ or $f(x) - -x$. Is this statement true for odd-dimensional spheres? (Prove or provide a counter-example.)