BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

2 June 2009

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3, A4.

Examiner 2: Real Analysis, questions B1, B2.

Examiner 3: Complex Analysis, questions B3, B4.

Examiner 4: Applied Mathematics, questions C1, C2, C3, C4.

Examiner 5: Geometry, questions D1, D2.

Examiner 6: Topology, questions D3, D4.

Time allowed: three hours.

Section A: Algebra

A1: Prove that every vector space has a bases. Hence determine, up to isomorphism, the groups G such that the automorphism group $\mathrm{Aut}(G)$ is trivial. (In this question, any Zorn's Lemma argument must be presented in full detail.)

A2: Let G be the group of permutations σ of $\mathbb{N}$ such that the support $\mathrm{supp}(\sigma) = \{i \in \mathbb{N} : \sigma(i) \neq i\}$ is finite. Find the normal subgroups of G. (You may assume, without proof, that the alternating group A_n is simple for $n \geq 5$.)

A3: (Parts (a) and (c) are especially easy, but they contain most of the weighting, because the purpose of the question is just to test basic knowledge.) Let E be the splitting field for $X^{16} - 1$ over $\mathbb{Q}$, and let $F = \mathbb{Q}[\sqrt{2}, i]$.

(a) 4 marks. Explicitly describe a minimal generating set for the group $G = \mathrm{Gal}(E/\mathbb{Q})$ and likewise for the group $H = \mathrm{Gal}(F/\mathbb{Q})$.

(b) 1 mark. In terms of the generators that you have given, explicitly describe the group epimorphism $G \to H$ that is associated with the inclusion $F \hookrightarrow E$.

(c) 4 marks. Draw the Hasse diagrams of the poset of intermediate subfields $\mathbb{Q} \leq L \leq F$ and the poset of intermediate subfields $\mathbb{Q} \leq K \leq E$. (You may specify the fields L and K in terms of the corresponding subgroups of the Galois groups; it is not necessary to specify them by means of generating sets over $\mathbb{Q}$.)

(d) 1 mark. Using parts (b) and (c), show that there are precisely 3 intermediate fields $\mathbb{Q} \leq K \leq E$ with the property that, if $\mathbb{Q} \leq K' \leq E$ and $K \cap F = K' \cap F$, then $K = K'$.

A4: For a finite poset P and a field F, let $F[P]$ be the F-algebra with basis $\{e_{x,y} : P \ni x < y \in P\}$ and with multiplication given by

$$e_{x,y}\, e_{y',z} = \begin{cases} e_{x,z} & \text{if } y = y', \\ 0 & \text{otherwise.} \end{cases}$$

(a) Let $J(F[P])$ denote the Jaconson radical of $F[P]$. In combinatorial terms, express the dimension $\dim_F(F[P]/J(F[P]))$, the number of isomorphism classes of simple $F[P]$-modules, and the Loewy length of $F[P]$. (Recall that the Loewy length n is defined by the condition $J^{n-1}(F[P]) > J^n(F[P]) = 0$.)

(b) Classify, up to isomorphism, the finite posets P such that $F[P]$ is a local ring.

Section B: Analysis

B1: Let $f \in L_p(\mu) \cap L_r(\mu)$ for $1 < p < r < \infty$. Show that $f \in L_q(\mu)$ for $p \le q \le r$.

B2: Let H be a Hilbert space and $T : H \longrightarrow H$ be a linear operator. Suppose $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$. Show that the operator T is bounded. (Hint: Use the Closed Graph Theorem or the Banah-Steinhaus Theorem.)

B3: Let $S \subset \mathbb{C}$ be a strip, that is, the open region between two parallel lines in the plane. Let $Q \subset \mathbb{C}$ be half of a disc, that is, the open region in a disc in the plane that lies on one side of a diameter. Construct a conformal map of S onto Q, writing the explicit functions when necessary.

B4: Suppose $R > 1$ and f is meromorphic in the disc $D(0, R)$ whose only singularity is a simple pole at c with $|c| = 1$. Let the power series expansion of f in the open unit disc be $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Prove that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = c$.

Section C: Applied Mathematics

C1: Let

$$e^{-t^2+2xt} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \,.$$

Find and prove the following for the Hermite Polynomials by using its generating function given above:
(a) $H_n(-x) = (-1)^n H_n(x)$, **(b)** $||H_n||$, **(c)** $H_n(0)$,
(d) $H_{n+1} - 2xH_n = 2nH_{n-1}$, **(e)** $H_n'' - 2xH_n' + 2nH_n = 0$.

C2: Find the extremals of the functional

$$J(y) = \int_0^1 [(y')^2 + x^2] \, dx$$

so that $y(0) = 0, \quad y(1) = 0$ with the constraint

$$\int_0^1 y^2 \, dx = 2 \,.$$

C3: Consider the integral equation

$$y(x) = f(x) + \lambda \int_0^1 (1 - k\,x\,s) y(s) ds$$

where $f(x)$ is a continuous function in $[0, 1]$, k is a constant and λ is the parameter of the integral equation. Solve the above integral equation by discussing all possible cases.

C4: Use the singular perturbation method to obtain a uniform approximate solution to the following boundary value problem. We assume that $0 < \varepsilon << 1$ and $0 < t < 1$.

$$\varepsilon y'' + (t+1)y' + y = 0, \quad y(0) = 0, \; y(1) = 1.$$

Section D: Geometry and Topology

D1: Show that the unit sphere S^2 in $\mathbb{R}^3$ can be given the structure of a complex manifold of dimension one, by constructing a holomorphic coordinate chart. Show that this manifold is bi-holomorphic to the complex projective line $\mathbb{P}^1$. Letting L be the line in $\mathbb{C}^2$ passing through the origin and the point $(\alpha, \beta) \neq (0,0)$, show to which point on S^2 it maps under the bi-holomorphic map you constructed above.

D2: For an ideal J in the polynomial ring $k[x_1, \ldots, x_n]$, where k is a field, we write $Z(J)$ for the set of all n-tuples $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ for which $f(a) = 0$ for all $f \in I$. And conversely for any subset $U \subset \mathbb{C}^n$ we write $I(U)$ for the set of all polynomials $f \in k[x_1, \ldots, x_n]$ which satisfy the condition that $f(a) = 0$ for all $a \in U$. We denote the radical of an ideal J by writing $\sqrt{J}$. Assume the weak nullstellensatz which says that if k is algebraically closed then for any proper ideal J we have $Z(J) \neq \emptyset$. Prove the strong form of nullstellensatz: For any ideal J we have $I(Z(J)) = \sqrt{J}$.

D3: **(a)** For a family $\{X_a : a \in A\}$ of topological spaces, define the product topology on $X = \prod_{a \in A} X_a$ and state Tychonoff's Theorem.

(b) Towards proving Tychonoff's Theorem, suppose that the family $\{X_a : a \in A\}$ is a counter-example. Show that there exists a family $\mathcal{C}$ of subsets of X such that the following conditions hold: $\bigcap_{C \in \mathcal{C}} \overline{C} = \emptyset$ where $\overline{C}$ denotes the closure of C; the intersection of finitely many elements of $\mathcal{C}$ is always non-empty; if $X \supseteq C' \supseteq C \in \mathcal{C}$ then $C' \in \mathcal{C}$; if $C_1, C_2 \in \mathcal{C}$ then $C_1 \cap C_2 \in \mathcal{C}$; if $C' \cap C \neq \emptyset$ for all $C \in \mathcal{C}$ then $C' \in \mathcal{C}$.

(c) Letting $\pi_a : X \to X_a$ be the projection, show that there exists an element $x_a \in \bigcap_{C \in \mathcal{C}} \pi_a(C)$.

(d) Show that $\pi_a^{-1}(U) \in \mathcal{C}$ for any neighbourhood U of x_a. Hence complete the proof of the theorem.

D4: **(a)** Given a finite simplicial complex K, and writing $|K|$ to denote the associated topological space, define the boundary map $C_n(K) \to C_{n-1}(K)$, where $C_n(K)$ is the $\mathbb{Z}$-module freely generated by the n-simplexes of K. (Deal carefully with the matter of the orientation of the n-simplexes for $n \geq 2$.)

(b) Define the homology $H_*(K) = H_*(|K|)$ in terms of the chain complex $C_*(K)$.

(c) Letting $\overline{K}_n$ be the finite simplicial complex associated with the n-simplex Δ_n, explain why $H_0(\overline{K}_n) \cong \mathbb{Z}$ and $H_n(\overline{K}_n) = 0$ for $n \geq 1$. (You may assume that the homology $H_*(K) = H_*(|K|)$ is a topological invariant of $|K|$.)

(d) For a natural number n, let us understand the **n-sphere** S^n to be the boundary of Δ_n. Specify a triangulation K_n of S^n with $K_n \leq \overline{K}_n$.

(e) Starting from the definitions in part (a) and (b), then arguing in a combinatorial way using parts (c) and (d), determine the homology $H_*(S^n)$.