BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

6 January 2009

Instructions:

- The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.
- Hand in up to SIX separate scripts, one script for each examiner, labelled as follows:

  $\aleph$: Algebra, questions A1, A2, A3, A4.

  $\mathbb{R}$: Real Analysis, questions B1, B2.

  $\mathbb{C}$: Complex Analysis, questions B3, B4.

  $\int$: Applied Mathematics, questions C1, C2, C3, C4.

  $\triangle$: Geometry, questions D1, D2.

  $\circlearrowleft$: Topology, questions D3, D4.

Time allowed: three hours.

Section A: Algebra

$\aleph$, **A1:** For the purposes of this question, we define a **Sylow p-subgroup** of a finite group G to be a p-subgroup S of G such that the index $|G : S|$ is not divisible by the prime p.

(a) State Sylow's Theorem.

(b) Show that the Sylow p-subgroups are precisely the maximal p-subgroups. (Hint: consider the conjugation action of a given p-subgroup on the set of Sylow p-subgroups.)

(c) Improve the conclusion of the enumeration part of Sylow's Theorem in the special case where $S \cap T = 1$ for any two distinct Sylow p-subgroups.

$\aleph$, **A2:** **(a)** State the Fundamental Theorem of Galois Theory (including clauses concerning normality, degrees of extensions and orders of subgroups).

(b) Let E be the splitting field for the polynomial $X^{30} + X^{29} + ... + X^2 + X + 1$ over $\mathbb{Q}$. You may assume, without proof, that this polynomial is irreducible. Consider an intermediate field $\mathbb{Q} \leq L \leq E$. Show that the extensions E/L and $L/\mathbb{Q}$ are Galois.

(c) Find the number of intermediate fields $\mathbb{Q} \leq L \leq E$.

$\aleph$, **A3:** Let $\mathcal{A}$ be a category, let I be a set and, for each $i \in I$, let A_i be an object in $\mathcal{A}$. Define the **sum** $\bigoplus_{i \in I} A_i$ and the **product** $\prod_{i \in I} A_i$. Hint: in the case $I = \{1, 2\}$, the definitions are indicated by the following diagram.

(a) Show that, when they exist, $\bigoplus_i A_i$ and $\prod_i A_i$ are unique up to isomorphism.

(b) Show that $\bigoplus_i A_i$ and $\prod_i A_i$ always exist in the category of abelian groups and in the category of sets.

(c) Letting $\mathcal{A}$ be the category of abelian groups, and letting A_1, A_2, ... be non-trivial finite abelian groups, show that $\bigoplus_{i=1}^{\infty} A_i \not\cong \prod_{i=1}^{\infty} A_i$. (Warning: of course, isomorphism is not the same thing as equality.)

(d) Letting $\mathcal{A}$ be the category of sets, and letting A_1, A_2 be non-empty finite sets, find a necessary and sufficient condition for the isomorphism $A_1 \oplus A_2 = A_1 \times A_2$.

$\aleph$, **A4:** Give three definitions (distinct but equivalent) of the term **semisimple ring**. For each of the following statements, give either a proof or a counter-example.
(i) Any subring of a semisimple ring is semisimple. (Any ring R is deemed to have a unity element 1_R.)
(ii) Any unital subring of a semisimple ring is semisimple. (A subring $S \leq R$ is said to be unital provided $1_S = 1_R$.)
(iii) Any quotient ring of a semisimple ring is semisimple.
(iv) Given a semisimple ring R, then the centre $Z(R)$ is semisimple.

Section B: Analysis

$\mathbb{R}$, **B1:** Let a measure μ be defined by the conditions $\mu\{n\} = 3^{-n}$ for $n \in \mathbb{N}$ and $\mu(E) = 0$ for every E with $E \cap \mathbb{N} = \emptyset$. Find $\displaystyle\lim_{k\to\infty} \int \frac{e^x \sin kx}{k^2 + x^2}\, d\mu$.

$\mathbb{R}$, **B2:** Let $A : X \longrightarrow l_p : f \mapsto \left(\int_0^1 x^n f(x)\, dx \right)_{n=0}^{\infty}$.

(a) Show that in the case $X = C[0,1]$ the operator A is well-defined and continuous if and only if $p > 1$. Find its norm for every $p > 1$.

(b) Show that in the case $X = L_q(0,1)$ with $q > 1$, the operator A is well-defined and continuous provided $p > q/(q-1)$.

$\mathbb{C}$, **B3:** Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disc, $a \in \mathbb{D}$, $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$, and $0 < r < 1$. Show that the image of the disc $r\mathbb{D} = \{ z \in \mathbb{C} : |z| < r \}$ under φ_a is also a disc lying in $\mathbb{D}$ with center $c = \dfrac{(1-r^2)\,a}{1-r^2|a|^2}$ and radius $R = \dfrac{(1-|a|^2)\,r}{1-r^2|a|^2}$.

$\mathbb{C}$, **B4:** Prove that there is **no** holomorphic function f defined on the annulus $A = \{ z \in \mathbb{C} : 1 < |z| < 3 \}$ satisfying $\left| \dfrac{(f(z))^2}{z} - 1 \right| < 1$ for all $z \in A$.

Section C: Applied Mathematics

$\int$, **C1:** Consider the sequence

$$h_n(x) = \begin{cases} 0 & \text{if } x \leq -1/n, \\ (nx+1)/2 & \text{if } -1/n \leq x \leq 1/n, \\ 1 & \text{if } x \geq 1/n. \end{cases}$$

(a) Prove that $h_n(x) \to \theta(x)$, where $\theta(x)$ is the step function, and

(b) $\dfrac{dh_n(x)}{dx} \to \delta(x)$ (thus we can formally write $\dfrac{d\theta(x)}{dx} = \delta(x)$.)

$\int$, **C2:** Let $y' + y - \varepsilon y^2 = 0$ for $t > 0$ with $y(0) = 1$. Find approximate solutions (up to second order perturbation for instance) and discuss whether the approximate solution is a uniformly valid solution.

$\int$, **C3:** Find the solution of the heat equation $u_t = u_{xx} + xe^{-t}$ with

$$u(0,t) = 0\ , \qquad u_x(1,t) + u(1,t) = 0\ , \qquad u(x,0) = 0\ .$$

$\int$, **C4:** Let $u \in C^2(D) \cap C^1(\overline{D})$, where D is a bounded domain. Show that:

(a) If $\nabla^2 u \leq 0$ in D and $u \geq 0$ on B, then $u \geq 0$ in D,

(b) If $\nabla^2 u \geq 0$ in D and $u \leq 0$ on B, then $u \leq 0$ in D.

Section D: Geometry and Topology

$\triangle$, **D1:** Let $\mathbb{P}^2$ be the projective plane over the complex numbers and let X and Y be some local coordinates around some point.

(a) Find the two intersection points in $\mathbb{P}^2$ of the parabola $Y = X^2$ with the line $X = 0$.

(b) Find the intersection point in $\mathbb{P}^2$ of the lines $2Y + 3X = 0$ and $2Y + 3X = 1$.

$\triangle$, **D2:** Let x and y be the coordinates in the affine plane $\mathbb{A}^2$ over a field $\mathbf{k}$. Show by direct calculation that the singularity of the cusp $y^2 = x^5$ can be resolved by a succession of blow-ups. Which properties of the field $\mathbf{k}$ are used?

$\circlearrowleft$, **D3:** State the Excision Axiom of the Eilenberg-Steenrod Axioms for a homology theory. Give a non-trivial example that illustrates how this axiom can be used to do calculations.

$\circlearrowleft$, **D4:** Calculate the homology groups $H_n(\mathbb{R}P^{2k+1}; \mathbb{Z}/3)$ for all $n \geq 0$ and for all $k \geq 0$. Here $\mathbb{R}P^{2k+1}$ denotes the real projective space of dimension $2k + 1$ and $\mathbb{Z}/3$ denotes the ring of integers mod 3.