BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

20 May 2008

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in FIVE separate scripts, one script for each examiner, labeled $\aleph$ (algebra), $\mathbb{R}$ (real analysis), $\mathbb{C}$ (complex analysis), $\int$ (applied mathematics), $\circlearrowleft$ (topology).

Time allowed: three hours.

Section A: Algebra

($\aleph$1) (a) State Sylow's Theorem.

(b) Show that, up to isomorphism, there is a unique group G with minimal order $|G|$ such that $|G|$ is odd and G is non-abelian.

($\aleph$2) (a) State the Fundamental Theorem of Galois Theory (including clauses concerning the order of the Galois group and the normal subgroups of the Galois group.)

(b) Let E be the splitting field for a polynomial f over a field F with characteristic zero. What is the maximum number of fields K such that $F \leq K \leq E$ and the extension K/F is normal when **(i)** f is a cubic? **(ii)** f is a quartic? (Hint: $1 \trianglelefteq V_4 \trianglelefteq A_4 \trianglelefteq S_4$.)

($\aleph$3) (a) State Zorn's Lemma.

(b) Show that any vector space has a basis. (Recall that a **basis** is defined to be a linearly independent spanning set. Hint: first deal with the case where the vector space has a finite spanning set.)

($\aleph$4) Construct the ordinary character table for the alternating group A_5. (If you wish, you may assume that A_5 is isomorphic to the rotation group of a dodecahedron.)

Section B: Analysis

($\mathbb{R}$1) Let $f(x) = x^\alpha \sin(x^\beta)$, $x \in (0,1)$, $\alpha, \beta \in \mathbb{R}$. Give examples of parameters α and β when the function f

(a) is Riemann integrable, but has not bounded variation,

(b) is improper Riemann integrable, but does not belong to $L(0,1)$.

($\mathbb{R}$2) Show that the operator $T : l_\infty \longrightarrow l_p : (x_n)_1^\infty \mapsto (x_n/n)_1^\infty$ is compact for any $1 < p \leq \infty$.

($\mathbb{C}$1) In each of the following, either write a holomorphic function on the unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ with the stated properties, or prove that no such function on $\mathbb{D}$ exists.

(a) $|h(z)| \leq 1$ for all $z \in \mathbb{D}$, $h\left(1 - \dfrac{1}{n^2}\right) = 0$ for $n = 2, 3, \ldots$, and h has no other zeros.

(b) $g(1/4) = 0$, $g(1/5) = 0$, and $g(0) = 1/10$.

($\mathbb{C}$2) Let f be a holomorphic function defined on the annulus $B = \{ z \in \mathbb{C} : 0 < |z| < 1 \}$ satisfying $\displaystyle\int_B |f(z)|^2 \, dA < \infty$, where dA is the area measure. Prove that f has a removable singularity at 0.
(*Hint:* Use polar coordinates and show that the unwanted coefficients in the Laurent expansion $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} c_n z^n$ are 0.)

Section C: Applied Mathematics

($\int$1) Let $u_m(x)$, $(m = 0, 1, 2, \cdots)$ be one of the classical orthonormal polynomial with weight function $w(x)$ and $x \in [a, b]$. Then prove that the sequence $h_n(x) = w \sum_{k=0}^{n} u_k(x)\, u_k(y)$, $(n = 1, 2, \cdots)$ where $y \in [a, b]$ defines a distribution over $(-\infty, \infty)$ called the Dirac δ-function $\delta(x - y)$. Prove that $\int_a^b \delta(x - y)\, f(x) dx = f(y)$ if $a < y < b$. It is equal to zero if $b < y < a$.

($\int$2) Use singular perturbation method to obtain a uniform approximate solution to the following problem. Assume $0 < \varepsilon << 1$ and $0 < t < 1$.

$$\varepsilon y'' + 2y' + e^y = 0, \quad y(0) = y(1) = 0\,.$$

($\int$3) Homogeneous wave equation with initial and boundary value problem. Let

$$u_{tt} - c^2\, u_{xx} = 0,$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \le x \le L$$
$$u(0, t) = u(L, t) = 0, \quad t \ge 0$$

Under what conditions on the data $\{f(x), g(x)\}$ the above problem is well-posed?.

($\int$4) Obtain the most general solution of the integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x + s)\, y(s) ds.$$

when $f(x) = x$ and $f(x) = 1$. Discuss all cases.

Section D: Geometry and Topology

(↺1) Let F_2 denote the free group in two generators, a and b. By considering the covering spaces of the one-point union of two circles (a figure-8), write explicit sets of generators (in terms of a and b) for all the index 3 subgroups of F_2 that are not normal in F_2.

(↺2) Let X be a metric space. If there exists a bijective function from X into the real numbers $\mathbb{R}$ then prove that there is a continuous function from X into $\mathbb{R}$ whose range is uncountable.