BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

8 January 2008

Instructions:

• The four sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in four separate scripts, one script for each section. (Each of the four scripts will be sent to a different examiner.)

Time allowed: three hours.

Section A: Algebra

(1) Recall that, for a group G, the Frattini subgroup $\Phi(G)$ is defined to be the intersection of all the maximal subgroups of G. Suppose that every maximal subgroup of G is normal and has prime index in G.

(a) Show that $G/\Phi(G)$ is abelian.

(b) Show that the order of any element of $G/\Phi(G)$ is a square-free number.

(c) Suppose, furthermore, that G is finitely generated. Show that $G/\Phi(G)$ is finitely generated and that the minimum number of generators for G is equal to the minimum number of generators for $G/\Phi(G)$.

(2) (a) State the Fundamental Theorem of Galois Theory (including clauses concerning the order of the Galois group and the normal subgroups of the Galois group.)

(b) Let E be the splitting field for $X^6 - 2$ over $\mathbb{Q}$. Show that the Galois group for E over $\mathbb{Q}$ is isomorphic to the dihedral group with order 12.

(c) How many intermediate fields $\mathbb{Q} \leq K \leq E$ are there?

(3) (a) State Zorn's Lemma.

(b) Let Λ be a ring with a unity element and let M be a Λ-module such that M is the sum of the simple modules of M. Give a full and detailed proof that M is a direct sum of simple modules. (Hint: let X be the set of simple submodules of M. Consider the subsets Y of X such that the sum $\sum_Y Y$ is a direct sum $\bigoplus_Y Y$.)

(4) Construct the ordinary character table for the group $\langle x, y : x^7 = y^3 = 1, yxy^{-1} = x^2\rangle$. (Hint: first show that 1, x, x^{-1}, y, y^2 is a full set of representatives for the conjugacy classes.)

Section B: Analysis

(1) Let $f \in L^p(\mathbb{R})$ for some p with $0 < p < \infty$. Put $E_y = \{\, x \in \mathbb{R} : |f(x)| > y \,\}$ and define $S(y) = |E_y|$, where $|\cdot|$ denotes the one-dimensional Lebesgue measure.

(a) Prove that S is a nondecreasing function of y and that $\lim_{y \to +\infty} S(y) = 0$.

(b) Prove that $\int_{\mathbb{R}} |f(x)|^p \, dx = p \int_0^\infty y^{p-1} \, S(y) \, dy$. (*Hint:* Start with a nice class of functions f, *or*, use Fubini theorem.)

(2) Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A bounded linear operator $T : H \to H$ is called compact if for every bounded sequence $\{x_n\}$ in H, there is a subsequence $\{x_{n_j}\}$ so that $\{T(x_{n_j})\}$ converges in norm in H. Equivalently, such a T is called compact if $\{x_n\}$ is a sequence that converges weakly in H, then $\{T(x_n)\}$ converges in norm in H.

(a) If in H a sequence $\{x_n\}$ converges weakly to x_0 and a sequence $\{y_n\}$ converges in norm to y_0, prove that $\lim_{n \to \infty} \langle x_n, y_n \rangle = \langle x_0, y_0 \rangle$.

(b) If T is a compact operator on H, prove that there is an $x_0 \in H$ with $\|x_0\| = 1$ such that $\sup_{\|x\|=1} \; = \langle T(x), x \rangle = \langle T(x_0), x_0 \rangle$.

(3) Let g be a branch of $\sqrt{z}$ on $\mathbb{C} \setminus (-\infty, 0]$ with $g(2) > 0$. Let h be a branch of $\sqrt{1-z}$ on $\mathbb{C} \setminus (-\infty, 1]$ with $h(2)$ purely imaginary. Define $f = gh$.

(a) Compute the limits $\lim_{y \to 0^-} g(x+iy)$, $\lim_{y \to 0^+} g(x+iy)$, and $\lim_{y \to 0^-} h(x+iy)$, $\lim_{y \to 0^-} h(x+iy)$ for any x with $0 < x < 1$, and show that f is not continuous across $(0,1)$. (First write down the intervals for the arguments of g and h.)

(b) Show, however, that f is continuous across $(-\infty, 0)$.

(c) Prove rigorously that f extends holomorphically to $\mathbb{C} \setminus [0,1]$.

(4) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a holomorphic function defined on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $\int_{\mathbb{D}} |f|^2 \, dA$ is finite, where dA is the area measure.

(a) Prove that $\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \leq C$ for some constant C. (Make sure you justify your steps.)

(b) Prove that $|f(z)| \leq \frac{C}{1 - |z|}$ for $z \in \mathbb{D}$.

(c) If $|f(z)| \leq \frac{M}{1 - |z|}$ for some constant M, show with a counterexample that it does not follow that $\int_{\mathbb{D}} |f|^2 \, dA$ is finite.

Section C: Applied Mathematics

(1) [MATH 543] Let $J(y) = \int_0^\pi (y'')^2\, dx$ with constraint $\int_0^\pi y^2 dx = 1$ and with the boundary conditions $y(0) = y''(0) = 0$, $y(\pi) = y''(\pi) = 0$. Find a solution for $y(x)$ which extremises the functional $J(y)$ with constraint.

(2) [MATH 543] Solve $u'' + u = f(x)$ with $u(0) = 0,\ u(a) = 0$. Discuss all possible cases.

(3) [MATH 544] Let

$$\frac{dx}{dt} = x - xy, \quad \frac{dy}{dt} = y + 2xy$$

Find all critical points and determine whether each critical point is asymptotically stable, stable, or unstable and classify it as to type.

(4) [MATH544] Consider

$$\nabla^2 u(r,\theta) = 0, \quad 0 \le r < a$$

$$\frac{\partial u(a,\theta)}{\partial r} + \alpha u(a,\theta) = f(\theta),$$

where f is a periodic function with period 2π and $\alpha > 0$ is a constant. Find the formal solution of the above equation. Find reasonable conditions on f so that the the formal solution is a solution of the above boundary value problem.

Section D: Geometry and Topology

(1) A topological space X is called *locally Hausdorff* if every neighborhood of every point $x \in X$ contains a Hausdorff subneighborhood of x. Prove or disprove:

(a) Every Hausdorff space is locally Hausdorff.

(b) Every locally Hausdorff space is Hausdorff.

(2) Let $\{A_\alpha\}$ be a family of closed subspaces of a topological space X, and let $A = \bigcap_\alpha A_\alpha$ be their intersection. Prove or disprove:

(a) If every finite intersection $\bigcap_i A_{\alpha_i}$ is path connected, then so is A.

(b) If each A_α is compact, then the statement (a) holds.