BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

27 November 2006

Instructions:

• The four sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• The six examiners are labelled α, β, γ, δ, ϵ, λ. Hand in separate scripts for each examiner. Thus, you are to hand in at most SIX scripts.

• Write your NAME and the LABEL OF THE EXAMINER on EVERY sheet that you hand in. (Each sheet should carry the label of exactly one examiner. It will not be possible for any sheet to be directed to more than one examiner.)

Failure to follow the above instructions may lead to loss of credit due to lack of information concerning the identity of the candidate or the identity of the appropriate examiner.

Good Luck!

Time allowed: four hours.

Section A: Algebra

(1α) Let $\mathcal{X}$ be a non-empty class of groups which is closed under isomorphism, subgroups and direct products. We mean to say that if G_1 and G_2 belong to $\mathcal{X}$, then every group isomorphic to G_1 belongs to $\mathcal{X}$, every subgroup of G_1 belongs to $\mathcal{X}$, and $G_1 \times G_2$ belongs to $\mathcal{X}$. Consider the following assertion: there exists a unique normal subgroup $O^{\mathcal{X}}(G)$ of G which is minimal subject to the condition that $G/O^{\mathcal{X}}(G)$ belongs to $\mathcal{X}$.

(a) Prove the assertion in the case where G is a finite group.

(b) Give an example of a group G and a class $\mathcal{X}$ such that the hypothesis on $\mathcal{X}$ is satisfied but the assertion fails.

(2α) Let K be a finite-degree extension field of $\mathbb{Q}$.

(a) State a necessary and sufficient condition for K to be a Galois extension of $\mathbb{Q}$. Using this criterion, show that there exists a finite-degree extension field E of K such that E is a Galois extension of $\mathbb{Q}$.

(b) Give an example where K is not a Galois extension of $\mathbb{Q}$. For this example, find a field E as in part (a). Up to isomorphism, determine the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ and determine the subgroup of $\mathrm{Gal}(E/\mathbb{Q})$ corresponding to K.

(3α) Recall that a module is said to be semisimple provided it is a direct sum of simple modules. Let $\Lambda = \prod_{i \in I} F_i$ as a direct product of fields. Write ${}_\Lambda\Lambda$ to denote Λ regarded as a left Λ-module by left multiplication. Using the definition of semisimplicity given at the beginning of the question, show that ${}_\Lambda\Lambda$ is semisimple if and only if the indexing set I is finite. [*Hint:* Consider the direct sum $\bigoplus_i F_i$.]

(4α) Construct the ordinary character table for D_{12}, the dihedral group with order 12.

Section B: Analysis

(1β) Find the limit $\lim_{n\to\infty} \int_0^n x^{-1/2}\,(1+n^2x^2)^{-1/3}\cos nx\,dx.$

(2β) Let X be the vector space of all functions on a set $K \subset \mathbb{R}$ with the topology τ defined by the seminorms $p_t(x) = |x(t)|,\ t \in K$. In terms of cardinality of the set K, give a criterion for the locally convex space (X,τ) to be:

(a) normable;

(b) metrizable.

(c) Let $K = [0,1]$. Is the set $A = \{x \in X : x(1/n) = 0,\ n \in \mathbb{N}\}$ open in the space (X,τ)?

(3γ) Let $D = \{\, z \in \mathbb{C} : 0 < |z| < 1 \,\}$ and let $\{\, p_n : n = 1,2,\dots \}$ be a sequence of distinct points in D with $p_n \to 0$ as $n \to \infty$. Suppose that f is holomorphic in $E = D \setminus \{\, p_n : n = 1,2,\dots \}$ and has a pole at each p_n (so that 0 is not an isolated singularity of f). Prove that $f(E)$ is dense in $\mathbb{C}$.

(4γ) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a holomorphic function in the unit disc $\mathbb{D} = \{\, z \in \mathbb{C} : |z| < 1 \,\}$.

Prove the following, where B, C, K, L are positive constants that do not depend on z or n.

(a) If $|f(z)| \le \dfrac{B}{1-|z|}$ for all $z \in \mathbb{D}$, then $|a_n| \le Cn$ for all $n = 1,2,\dots$.

(b) If $|a_n| \le K$ for all $n = 0,1,2,\dots$, then $|f(z)| \le \dfrac{L}{1-|z|}$ for all $z \in \mathbb{D}$.

Section C: Applied Mathematics

(1δ) Solve the boundary value problem
$u'' + \lambda u = 0,$
$u(0) = 0, \quad \cos\beta\, u(1) + \sin\beta u'(1) = 0.$

(2δ) Let $J(y) = \int_0^\pi y''^2 dx$ with the constraint $\int_0^\pi y^2 dx = 1$ and with the boundary conditions $y(0) = y''(0) = 0$, $y(\pi) = y''(\pi) = 0$. Find the function extremizing the functional $J(y)$.

(3δ) Show that there exists at most one solution of the BVP

$$u_t = \sigma u_{xxx}, \quad 0 < x < L, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 \le x \le L$$

$$u(0,t) = u_x(0,t) = 0, \quad u(L,t) = 0, \quad t \ge 0$$

and solve it by means of separation of variables. [*Hint*: Use the energy functional as $E(t) = \frac{1}{2}\int_0^L u^2 dx$].

(4δ) Find the solution of

$$y(x) = f(x) + \lambda \int_a^b e^{x-s}\, y(s) ds\,.$$

Section D: Geometry and Topology

(1ϵ) (a) Show that two affine algebraic varieties are isomorphic if and only if their coordinate rings are isomorphic.

(b) Show that this fails for projective varieties.

(c) Show that if an affine variety is isomorphic to a projective variety then it is a single point.

(2ϵ) (a) Find the Hilbert polynomial of a hypersurface of degree d in projective n space.

(b) Describe how you would find the Hilbert polynomial of a complete intersection in projective space.

(3λ) Prove or disprove the following statement: A finite cartesian product of connected spaces is connected.

(4λ) Let S^2 denote the 2-sphere and let p, q, r be distinct points on S^2. Let X be the identification space obtained by identifying p, q, r to a single point $x \in X$.

(a) Find a CW-complex structure for X.

(b) Compute the homology groups of X.