BILKENT UNIVERSITY
PhD QUALIFYING EXAM
IN MATHEMATICS

21 August 2006

Instructions:

- Attempt at most TWO questions from each of the four sections.
- Hand in four separate scripts, one script for each section.
- Write your name and the section letter on each sheet that you hand in.

Failure to follow the instructions will cause inconvenience and may cause marking omissions.

Time allowed: three hours.

Section A: Algebra

(1) Let p be a prime. How many isomorphism classes of groups G are there such that $|G| = p^3$ and every non-identity element of G has order p?

(2) (a) Give an example of a field K such that $\mathrm{Gal}(K/\mathbb{Q}) \cong C_6$. How many strictly intermediate fields $\mathbb{Q} < I < K$ are there?

(b) Give an example of a field L such that $\mathrm{Gal}(L/\mathbb{Q}) \cong S_6$. How many strictly intermediate fields $\mathbb{Q} < J < L$ are there?

(3) Assuming the existence of the algebraic closure of a given field, prove the uniqueness.

(4) Construct the ordinary character table for the group $A_4 \times C_2$.

Section B: Analysis

(1) (a) Let f be a holomorphic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ function such that $f'(1/n) = f(1/n)$ for $n = 2, 3, \dots$. Show that f is an entire function.

(b) Is there a non-zero holomorphic in $\mathbb{D}$ function f with $|f(1/n)| < 2^{-n}$ for all $n \geq 2$?

(2) Determine the largest open set to which the function $f(z) = \sum_1^\infty z^n/n^2$ can be analytically continued.

(3) Determine which of the following statements are true or false. If false, give a counter-example.

(a) If $f_n \longrightarrow f$ in $L_2(X, \mu)$ then $f_n \xrightarrow{\mu} f$.

(b) If $f_n \longrightarrow f$ in $L_1(\mathbb{R})$ and $f_n \longrightarrow g$ in $L_2(\mathbb{R})$ then $f \overset{\text{æ}}{=} g$.

(c) If $f_n \xrightarrow{au} f$ and $|f| \leq g$ and $g \in L_2(X, \mu)$ then $f_n \to f$ in $L_1(X, \mu)$.

(d) If $f_n \xrightarrow{\mu} f$ and $\lambda << \mu$ then $f_n \xrightarrow{\lambda} f$.

(4) Show that the unit sphere in a Hilbert space is not weakly compact.

Section C: Applied Mathematics

(1) Use the method of Frobenius to obtain the general solution to the differential equation $z^2u'' - 2zu' + (2 - z^2)u = 0$, valid near $z = 0$.

(2) Recall that the Hermite polynomials H_n have generating function

$$e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x)t^n/n! \ .$$

Show that:

(a) $H_n(-x) = (-1)^n H_n(x)$,

(b) $H_{n+1} - 2xH_n = 2nH_{n-1}$,

(c) $H_n'' - 2xH_n' + 2nH_n = 0$.

(3) Find the formal solution to $u_t = au_{xx}$ where $a > 0$, $t > 0$, $x \in [0, L]$ with $u(0, t) = u(L, t)$. Express the solutions in terms of the constants α_n where

$$u(x, 0) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x/L).$$

(You do not need to discuss the hypotheses under which the formal solution is valid.)

(4) Let D be a bounded normal domain with boundary B. Consider the Dirichlet problem

$$\nabla^2 u = h \text{ in } D \quad \text{and} \quad u = f \text{ on } B$$

where h and f are continuous functions in D and on B. Recall that the Green's function $G(x, y)$ is given by

$$\nabla_y^2 G(x, y) = \delta(x - y) \text{ in } D \quad \text{and} \quad G(x, y) = 0 \text{ for } y \in B.$$

(a) Without proof, express u in terms of G, h, f.

(b) Prove that $G(x, y) = G(y, x)$ for $x, y \in D$.

Section D: Topology and Geometry

(There is no geometry in this paper)

(1) Show that, in any topological space X, each quasi-component is a union of components, and each component is a union of path-components. (Recall that the *quasi-component* of a point $x \in X$ is the intersection of all the open-closed subsets of X containing x.)

(2) Let $K \subseteq \mathbb{R}^3$ be the trefoil knot (which can be thought of as the circle embedded in $\mathbb{R}^3$ via $f \mapsto (2\cos 2f + \cos 3f, 2\sin 2f + \cos 3f, \sin 3f)$ for $f \in [0, 2\pi]$; in other words, it wraps the standard torus two times along the parallels and three times along the meridians). Find the homology groups of the complement $\mathbb{R}^3 - K$.