An inversion formula for the primitive idempotents of the trivial source algebra

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Abstract

Formulas for the primitive idempotents of the trivial source algebra, in characteristic zero, have been given by Boltje and Bouc–Thévenaz. We shall give another formula for those idempotents, expressing them as linear combinations of the elements of a canonical basis for the integral ring. The formula is an inversion formula analogous to the Gluck–Yoshida formula for the primitive idempotents of the Burnside algebra. It involves all the irreducible characters of all the normalizers of p-subgroups. As a corollary, we shall show that the linearization map from the monomial Burnside ring has a matrix whose entries can be expressed in terms of the above Brauer characters and some reduced Euler characteristics of posets.

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1 Introduction

The Burnside ring $B(G)$ of a finite group G has a basis consisting of the isomorphism classes of transitive G-sets. Extending to coefficients in $\mathbb{Q}$, the isomorphism classes of transitive G-sets still comprise a basis for $\mathbb{Q}B(G)$. The primitive idempotents of $\mathbb{Q}B(G)$ comprise another basis for $\mathbb{Q}B(G)$. It is straightforward to express each element of the former basis as a linear

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combination of elements of the latter basis. The matrix associated with those linear combinations was studied by Burnside, who called it the table of marks for G. Gluck [Glu81] and Yoshida [Yos83, Section 3] inverted the table of marks, expressing each primitive idempotent as a linear combination of isomorphism classes of transitive G-sets. Boltje [Bol, Section 3] gave similar inversion formulas for the primitive idempotents of various monomial Burnside rings and character rings.

Let $\mathbb{F}$ be a field with prime characteritsic p and let $\mathbb{K}$ be a field of characteristic zero. Throughout, we shall assume that $\mathbb{K}$ and $\mathbb{F}$ are sufficiently large in the sense that they own enough p'-th roots of unity. Fixing an isomorphism between a sufficiently large group of p'-th roots of unity in $\mathbb{K}$ and a group of p'-th roots of unity in $\mathbb{F}$, we can understand that, for modules of all the group algebras over $\mathbb{F}$ that come into consideration, the Brauer characters have values in $\mathbb{K}$.

Recall, given a finite-dimensional $\mathbb{F}G$-module M, then every indecomposable direct summand of M has a trivial source if and only if M has a basis stabilized by a Sylow p-subgroup of G. When those equivalent conditions hold, we call M a **trivial source** $\mathbb{F}G$-module (or a **p-permutation** $\mathbb{F}G$-module). The **trivial source ring** $T_{\mathbb{F}}(G)$ of $\mathbb{F}G$, (also called the **p-permutation ring** of $\mathbb{F}G$), is defined to be the abelian group generated by the isomorphism classes $[M]$ of trivial source $\mathbb{F}G$-modules M, subject to the relations $[M] + [M'] = [M \oplus M']$. We make $T_{\mathbb{F}}(G)$ become a ring with multiplication coming from tensor product, $[M][M'] = [M \otimes_{\mathbb{F}} M']$. For an introduction to the theory of trivial source modules and the trivial source ring, see Section 2 or, for more detail, Bouc–Thévenaz [BT10, Section 2].

Let P be a p-subgroup of G. Write $\overline{N}_G(P) = N_G(P)/P$. Let ϕ be an irreducible Brauer character of $\mathbb{F}\overline{N}_G(P)$. Write E_ϕ for the indecomposable projective $\mathbb{F}\overline{N}_G(P)$-module such that the simple head $E_\phi/J(E_\phi)$ has Brauer character ϕ. We define an induced and inflated $\mathbb{F}G$-module

$$N_{P,\phi} = {}_G\mathrm{Ind}_{N(P)}\mathrm{Inf}_{\overline{N}(P)}(E_\phi) \ .$$

In Section 2 we shall show that, as P and ϕ vary, the isomorphism classes having the form $[N_{P,\phi}]$ comprise a $\mathbb{Z}$-basis for $T_{\mathbb{F}}(G)$ and a $\mathbb{K}$-basis for $\mathbb{K}T_{\mathbb{F}}(G)$. We shall call this basis the **canonical basis** for $T_{\mathbb{F}}(G)$. Also in Section 2, we shall also give a citation for the well-known fact that the primitive idempotents of $\mathbb{K}T_{\mathbb{F}}(G)$ comprise a $\mathbb{K}$-basis for $\mathbb{K}T_{\mathbb{F}}(G)$.

Formulas for the primitive idempotents of $\mathbb{K}T_{\mathbb{F}}(G)$ were given by Boltje [Bol, 3.6] and Bouc–Thévenaz [BT10, 4.12]. Our main result, Theorem 2.4, asserts a different formula which expresses each primitive idempotent as a linear combination of the elements of the canonical basis. Our formula, involving irreducible Brauer characters of $\mathbb{F}\overline{N}_G(P)$, is an inversion formula in the sense that it is analogous to the inversion formulas of Gluck and Yoshida. We shall prove it, in Section 3, by considering a matrix analogous to the table of marks.

The monomial Burnside ring $B_{\mathbb{F}}(G)$ has a $\mathbb{Z}$-basis consisting of the isomorphism classes of transitive $F^\times$-fibred G-sets, where $F^\times = \mathbb{F} - \{0\}$. Theorem 4.1 gives a formula for the matrix representing the linearization map $\mathrm{lin}_G : B_{\mathbb{F}}(G) \to T_{\mathbb{F}}(G)$ with respect to the above basis of $B_{\mathbb{F}}(G)$ and the canonical basis for $T_{\mathbb{F}}(G)$. The formula again involves irreducible Brauer characters of $\mathbb{F}\overline{N}_G(P)$.

2 Statement of the idempotent formula

After briefly reviewing some well-known material on the primitive idempotents of the trivial source algebra $\mathbb{K}T_{\mathbb{F}}(P)$, we shall state a formula for those idempotents and deduce a corollary.

Let P be a p-subgroup of G and let M be a trivial source $\mathbb{F}G$-module. We write M^P to denote the subspace of P-fixed elements of M. For $Q \leq P$, we define the relative trace map to be the linear map

$$\mathrm{tr}_Q^P \,:\, M^Q \ni m \mapsto \sum_{uQ \subseteq P} {}^u m \in M^P \,.$$

We define the **Brauer quotient** of M to be the $\mathbb{F}\overline{N}_G(P)$-module

$$M[P] = M^P / \sum_{Q<P} \mathrm{tr}_Q^P(M^Q) \,.$$

Since M is a trivial source module, it has a P-stable basis Ω. Writing Ω^P for the set of P-fixed elements of Ω, then $M[P]$ can be identified with the $\mathbb{F}$-span $\mathbb{F}\Omega^P$ of Ω^P.

Let $\mathcal{C}(G)$ be the set of pairs (P, ϕ) where P is a p-subgroup of G and ϕ is an irreducible Brauer character of $\mathbb{F}\overline{N}_G(P)$. We allow G to act on $\mathcal{C}(G)$ such that ${}^g(P, \phi) = ({}^gP, {}^g\phi)$ for $g \in G$. For each $(P, \phi) \in \mathcal{C}(G)$, we define $M_{P,\phi}^G$ to be the indecomposable $\mathbb{F}G$-module, unique up to isomorphism, such that $M_{P,\phi}^G$ has vertex P and $M_{P,\phi}^G$ is in Green correspondence with the inflated $\mathbb{F}N_G(P)$-module ${}_{N(P)}\mathrm{Inf}_{\overline{N}(P)}(E_\phi)$. Plainly, $M_{P,\phi}^G$ is a trivial source $\mathbb{F}G$-module and the isomorphism class $[M_{P,\phi}^G]$ depends only on the G-orbit of (P, ϕ). Using Bouc–Thévenaz [BT10, 2.7], it is straightforward to show that $M_{P,\phi}^G[P] \cong E_\phi$.

Proposition 2.1. (See [BT10, 2.9].) *As (P, ϕ) runs over representatives of the G-orbits of $\mathcal{C}(G)$, the isomorphism classes $[M_{P,\phi}^G]$ comprise a $\mathbb{Z}$-basis for $T_\mathbb{F}(G)$.*

Let $\mathcal{E}(G)$ denote the set of pairs $(Q, [s])$ where Q is a p-subgroup of G and $[s]$ is the conjugacy class of a p'-element of $\overline{N}_G(Q)$. We allow G to act on $\mathcal{E}(G)$ such that ${}^g(Q, [s]) = ({}^gQ, {}^g[s])$. For each $(Q, [s]) \in \mathcal{E}(G)$, we define a linear map

$$\epsilon_{Q,s}^G \,:\, \mathbb{K}T_\mathbb{F}(G) \to \mathbb{K}$$

such that, given a trivial source $\mathbb{F}G$-module M, then $\epsilon_{Q,s}^G[M]$ is the value, at s, of the Brauer character of $M[Q]$. It is easy to show that $\epsilon_{Q,s}^G$ depends only on the G-orbit of $(Q, [s])$. Part of [BT10, 2.18] asserts that $\epsilon_{Q,s}^G$ is an algebra map.

Proposition 2.2. (See [BT10, 2.18, 2.19].) *For each G-orbit of elements $(Q, [s]) \in \mathcal{E}(G)$, there exists a unique primitive idempotent $e_{Q,s}^G$ of $\mathbb{K}T_\mathbb{F}(G)$ such that $\epsilon_{Q,s}^G(e_{Q,s}^G) = 1$. Furthermore,*

$$\mathbb{K}T_\mathbb{F}(G) = \bigoplus_{(Q,[s]) \in_G \mathcal{E}(G)} \mathbb{K}\, e_{Q,s}^G$$

as a direct sum of algebras isomorphic to $\mathbb{K}$, the notation indicating that $(Q, [s])$ runs over representatives of the G-orbits of $\mathcal{E}(G)$.

By the two propositions above, we can write

$$[M_{P,\phi}^G] = \sum_{(Q,[s]) \in_G \mathcal{E}(G)} m_G(Q, s; P, \phi)\, e_{Q,s}^G \,, \qquad e_{Q,s}^G = \sum_{(P,\phi) \in_G \mathcal{C}(G)} m_G^{-1}(P, \phi; Q, s)\, [M_{P,\phi}^G] \,.$$

Some partial results about the coefficients $m_G(Q, s; P, \phi), m_G^{-1}(P, \phi; Q, s) \in \mathbb{K}$ will be given in the next section, but we do not have an explicit general formula for them. Of course,

$$m_G(Q, s; P, \phi) = \epsilon_{Q,s}^G[M_{P,\phi}^G]$$

but this equality is not explicit enough to admit much manipulation. That is why we shall focus on a different $\mathbb{Z}$-basis for $T_{\mathbb{F}}(G)$. Observe that the isomorphism class $[N^G_{P,\phi}]$ depends only on the G-orbit of (P, ϕ).

Proposition 2.3. *As (P, ϕ) runs over representatives of the G-orbits of $\mathcal{C}(G)$, the isomorphism classes $[N^G_{P,\phi}]$ comprise a $\mathbb{Z}$-basis for $T_{\mathbb{F}}(G)$.*

Proof. By the Green correspondence, $N^G_{P,\phi}$ is the direct sum of $M^G_{P,\phi}$ and an $\mathbb{F}G$-module whose indecomposable direct summands all have vertices strictly contained in P. The assertion now follows from Proposition 2.1. □

By Propositions 2.2 and 2.3, we can write

$$[N^G_{P,\phi}] = \sum_{(Q,[s]) \in_G \mathcal{E}(G)} n_G(Q, s; P, \phi)\, e^G_{Q,s} \, , \qquad e^G_{Q,s} = \sum_{(P,\phi) \in_G \mathcal{C}(G)} n^{-1}_G(P, \phi; Q, s)\, [N^G_{P,\phi}]$$

with $n_G(Q, s; P, \phi), n^{-1}_G(P, \phi; Q, s) \in \mathbb{K}$. Observe that

$$n_G(Q, s; P, \phi) = \epsilon^G_{Q,s}[N^G_{P,\phi}] \, .$$

We shall give an explicit formula for $n_G(Q, s; P, \phi)$ in Proposition 3.4 and an explicit formula for $n^{-1}_G(P, \phi; Q, s)$ in Theorem 3.6. As we shall prove in Section 3, the latter formula is equivalent to the next theorem.

To state the theorem, we need some more notation. Recall that any finite poset $\mathcal{P}$ can be associated with a simplicial complex whose simplices are the chains in $\mathcal{P}$. We define the **reduced Euler characteristic** of $\mathcal{P}$, denoted $\widetilde{\chi}(\mathcal{P})$, to be the reduced Euler characteristic of the simplicial complex. The **Möbius function** $\mu : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$ is defined to be the function such that, given $x, y \in \mathcal{P}$, then $\mu(x, x) = 1$; if $x \not\leq y$ then $\mu(x, y) = 0$; if $x < y$, writing $(x, y)_{\mathcal{P}} = \{z \in P : x < z < y\}$, then $\mu(x, y) = \widetilde{\chi}((x, y)_{\mathcal{P}})$. It is well-known that, for $x < y$, we have a recurrence relation $\sum_z \mu(x, z) = 0 = \sum_z (z, y)$ summed over z such that $x \leq z \leq y$. We shall be making use of that recurrence relation in the next section.

Let $\mathcal{S}_p(G)$ denote the G-poset of p-subgroups of G. For $g \in G$, the $\langle g \rangle$-fixed subposet $\mathcal{S}_p(G)^{\langle g \rangle}$ consists of those p-subgroups R such that $g \in N_G(R)$. We write μ_g to denote the Möbius function of $\mathcal{S}_p(G)^{\langle g \rangle}$.

Theorem 2.4. *Given $(Q, [s]) \in \mathcal{E}(G)$, then*

$$e^G_{Q,s} = \frac{1}{|N_G(Q)|} \sum_{(P,\phi) \in \mathcal{C}(G),\, gP \in \overline{N}_G(P)_{p'} \,:\, P \leq Q,\, g \in N_G(Q),\, gQ \in [s]} |P|\, \phi(g^{-1}P)\, \mu_g(P, Q)\, [N^G_{P,\phi}]$$

where $\overline{N}_G(P)_{p'}$ denotes the set of p'-elements of $\overline{N}_G(P)$.

The formula makes sense because the conditions $P \leq Q$ and $g \in N_G(Q)$ imply that the p'-element $gP \in \overline{N}_G(P)_{p'}$ determines the p'-element $gQ \in \overline{N}_G(Q)_{p'}$.

Lemma 2.5. *Given p-subgroups $P, Q \leq G$ and $g \in N_G(P, Q)$ such that $\mu_g(P, Q) \neq 0$, then $\Phi(Q) \leq P \leq Q$ where $\Phi(Q)$ denotes the Frattini subgroup of Q. So the sum in Theorem 2.4 can be taken over only those indices such that $\Phi(Q) \leq P \leq Q$.*

Proof. We apply the technique of conical contraction of posets, described in Benson [Ben91, 6.4.5, 6.4.6]. The condition $\mu_g(P,Q) \neq 0$ plainly implies that $P \leq Q$. For a contradiction, suppose that $\Phi(Q) \not\leq P$. Let $\mathcal{R}$ be the open interval (P,Q) in $\mathcal{S}_p(G)^{\langle g \rangle}$. Then $\mathcal{R}$ admits a contraction given by $R \mapsto R\Phi(Q) \mapsto P\Phi(Q)$ for $R \in \mathcal{R}$. This contradicts the condition that $\mu_g(P,Q) \neq 0$. □

Corollary 2.6. *Let* $(Q,[s]) \in \mathcal{E}(G)$ *and let*

$$I = \sum_{P\,:\,\Phi(Q) \leq P \leq Q} {}_G\mathrm{Ind}_{N_G(P)}(\mathbb{K}T_{\mathbb{F}}(N_G(P)))\ .$$

Then I *is an ideal of* $\mathbb{K}T_{\mathbb{F}}(G)$ *and* $e^G_{Q,s} \in I$.

Proof. By the Frobenius relations, I is an ideal. By Lemma 2.5, $e^G_{Q,s} \in I$. □

3 Proof of the idempotent formula

Let $\mathcal{M}$ be the matrix such that the rows are indexed by the G-orbits of elements $(Q,[s]) \in \mathcal{E}(G)$, the columns are indexed by the G-orbits of elements $(P,\phi) \in \mathcal{C}(G)$ and the $((Q,[s]),(P,\phi))$ entry is $m_G(Q,s;P,\phi)$. Let $\mathcal{N}$ be the matrix with the same indexing of rows and columns and with $((Q,[s]),(P,\phi))$ entry $n_G(Q,s;P,\phi)$. We shall be making a study of the matrices $\mathcal{M}$, $\mathcal{M}^{-1}$, $\mathcal{N}$, $\mathcal{N}^{-1}$.

Lemma 3.1. *Let* $(P,\phi) \in \mathcal{C}(G)$ *and* $(Q,[s]) \in \mathcal{E}(G)$.
(1) *If* $Q \not\leq_G P$*, then* $m_G(Q,s;P,\phi) = n_G(Q,s;P,\phi) = 0$.
(2) *If* $P \not\leq_G Q$*, then* $m_G^{-1}(P,\phi;Q,s) = n_G^{-1}(P,\phi;Q,s) = 0$.

Proof. The $\mathbb{F}G$-module $M^G_{P,\phi}$ is a direct summand of $N^G_{P,\phi}$ which is a direct summand of the permutation $\mathbb{F}G$-module $\mathbb{F}G/P$. If $Q \not\leq_G P$, then $\mathbb{F}G/P[Q] = 0$, hence $M^G_{P,\phi}[Q] = N^G_{P,\phi}[Q] = 0$. Part (1) follows. Thus, listing the conjugacy classes of p-subgroups of G in non-decreasing order, and regarding $\mathcal{M}$ and $\mathcal{N}$ as arrays of submatrices with rows indexed by the p-subgroup Q and columns indexed by the p-subgroup P, then $\mathcal{M}$ and $\mathcal{N}$ have an upper triangular form. The inverses $\mathcal{M}^{-1}$ and $\mathcal{N}^{-1}$ have the same upper triangular form, hence part (2). □

Let $\widehat{\phi}$ denote the Brauer character of the indecomposable projective $\mathbb{F}\overline{N}_G(P)$-module E_ϕ.

Proposition 3.2. *Given a p-subgroup P of G, an irreducible Brauer character ϕ of $\mathbb{F}\overline{N}_G(P)$ and a p'-element s of $\overline{N}_G(P)$, then*
(1) $m_G(P,s;P,\phi) = n_G(P,s;P,\phi) = \widehat{\phi}(s)$,
(2) $m_G^{-1}(P,\phi;P,s) = n_G^{-1}(P,\phi;P,s) = \phi(s^{-1})/|C_{\overline{N}_G(P)}(s)|$.

Proof. Part (1) holds because $M^G_{P,\phi}[P] \cong N^G_{P,\phi}[P] \cong E_\phi$. For part (2), we use the orthogonality relation for irreducible Brauer characters, which can be found in Feit [Fei82, 4.3.3]. Fixing P, let $\mathcal{L}$ be the matrix such that the rows are indexed by the conjugacy classes $[s]$ of p'-elements s of $\overline{N}_G(P)$, the columns are indexed by the irreducible Brauer characters ϕ of $\mathbb{F}\overline{N}_G(P)$ and the $([s],\phi)$ entry is $\widehat{\phi}(s)$. By part (1), $\mathcal{L}$ is a submatrix of $\mathcal{M}$ and $\mathcal{N}$. Since $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M}^{-1}$, $\mathcal{N}^{-1}$ all have an upper triangular form in the sense described in the proof of Lemma 3.1, $\mathcal{L}^{-1}$ is a submatrix of $\mathcal{M}^{-1}$ and $\mathcal{N}^{-1}$. By the orthogonality relation, $\mathcal{L}^{-1}$ has $(\phi,[s])$ entry

$$\phi(s^{-1})|[s]|/|\overline{N}(P)| = \phi(s^{-1})/|C_{\overline{N}(P)}(s)|\ .$$ □

We have nothing further to say about the matrices $\mathcal{M}$ and $\mathcal{M}^{-1}$. Our concern, for the rest of this section, will be with the matrices $\mathcal{N}$ and $\mathcal{N}^{-1}$. We regard $\mathcal{N}$ as an analogue of the table of marks. Our next main task is to find a general formula for the entries of $\mathcal{N}$.

Lemma 3.3. *Let P and Q be p-subgroups of G. Let E be a projective $\mathbb{F}\overline{N}_G(P)$-module and let $N = {}_G\mathrm{Ind}_{N_G(P)}\mathrm{Inf}_{\overline{N}_G(P)}(E)$. Then*

$$N[Q] = \bigoplus_{N_G(Q)gN_G(P) \subseteq G \,:\, Q \leq {}^gP} {}_{\overline{N}_G(Q)}\mathrm{Def}_{N_G(Q)}\mathrm{Ind}_{N_G(Q)\cap {}^gN_G(P)}({}^gE)$$

where ${}^gE = {}_{N_G(Q)\cap {}^gN_G(P)}\mathrm{Con}^g_{N_G(Q)^g \cap N_G(P)}\mathrm{Res}_{N_G(P)}\mathrm{Inf}_{\overline{N}_G(P)}(E)$.

Proof. As a direct sum of $\mathbb{F}Q$-modules,

$$N = \bigoplus_{QgN(P) \subseteq G} N_g$$

where $N_g = \mathbb{F}QgN(P) \otimes_{\mathbb{F}N(P)} E$. We have

$$N_g \cong {}_Q\mathrm{Ind}_{Q \cap {}^gN(P)}\mathrm{Res}_{{}^gN(P)}(\mathbb{F}gN(P) \otimes_{\mathbb{F}N(P)} E) \; .$$

So if $Q \not\leq {}^gN(P)$ then $N_g[Q] = 0$. Suppose that $Q \leq {}^gN(P)$. Then

$$N_g = {}_Q\mathrm{Res}_{{}^gN(P)}(\mathbb{F}gN(P) \otimes_{\mathbb{F}N(P)} E) \; .$$

Let $R = Q^g.P/P$. Since R is a p-subgroup of $\overline{N}(P)$ and E is a direct summand of the regular $\mathbb{F}\overline{N}(P)$-module, E has a basis B_g upon which R acts fixed-point-freely. The basis $g \otimes B_g$ of N_g is Q-stable. If $Q \not\leq {}^gP$, then R is non-trivial and every Q-orbit of $g \otimes B_g$ is non-singleton, hence $N_g[Q] = 0$. On the other hand, if $Q \leq {}^gP$, then R is trivial and Q fixes $g \otimes B_g$, hence we can make an identification $N_g[Q] = N_g$. Thus, we can make an identification

$$N[Q] = \bigoplus_{QgN(P) \subseteq G \,:\, Q \leq {}^gP} N_g = \bigoplus_{N(Q)gN(P) \subseteq G \,:\, Q \leq {}^gP} \mathbb{F}N(Q)gN(P) \otimes_{\mathbb{F}N(P)} E \; .$$

As an $\mathbb{F}N_G(Q)$-module fixed by Q, we have

$$\mathbb{F}N(Q)gN(P) \otimes_{\mathbb{F}N(P)} E \cong {}_{N(Q)}\mathrm{Ind}_{N(Q)\cap {}^gN(P)}({}^gE) \; . \qquad \square$$

Proposition 3.4. *Let $(Q, [s]) \in \mathcal{E}(G)$ and $(R, \psi) \in \mathcal{C}(G)$. Write $s = tQ$ with $t \in N_G(Q)$. Then*

$$n_G(Q, s; R, \psi) = \sum_{g \in G \,:\, {}^gQ \leq R,\, {}^gt \in N_G(R)} \widehat{\psi}({}^gt.R)/|N_G(R)| \; .$$

Proof. Abusing notation, writing ${}^g\widehat{\psi}$ to denote the conjugate by g of a restriction of an inflation of $\widehat{\psi}$, we have

$$\big({}_{N(Q)}\mathrm{ind}_{N(Q)\cap {}^gN(R)}({}^g\widehat{\psi})\big)(t) = \sum_{y \in N(Q) \,:\, {}^yt \in {}^gN(R)} {}^g\widehat{\psi}({}^yt)/|N(Q) \cap {}^gN(R)| \; .$$

So, by Lemma 3.3,

$$n_G(Q, s; R, \psi) = \sum_{N(Q)gN(R) \subseteq G,\, y \in N(Q) \,:\, Q \leq {}^gR,\, {}^yt \in {}^gN(R)} {}^g\widehat{\psi}({}^yt)/|N(Q) \cap {}^gN(R)|$$

$$= \sum_{g\in G,\, y\in N(Q)\,:\, Q\leq {}^gR,\, {}^yt\in {}^gN(R)} {}^g\widehat{\psi}({}^yt)/|N(Q)||N(R)|\ .$$

Making the substitution $f = g^{-1}$ and noting that ${}^g\psi({}^yt) = \psi({}^{fy}t)$, we have

$$n_G(Q,s;R,\psi) = \sum_{f\in G,\, y\in N(Q)\,:\, {}^fQ\leq R,\, {}^{fy}t\in N(R)} \widehat{\psi}({}^{fy}t.R)/|N(Q)||N(R)|$$

$$\sum_{fN(Q)\subseteq G,\, y\in N(Q)\,:\, {}^{fy}Q\leq R,\, {}^{fy}t\in N(R)} \widehat{\psi}({}^{fy}t)/|N(R)|\ . \qquad \square$$

We shall be making use of the following more complicated formula for $n_G(Q,s;P,\phi)$.

Proposition 3.5. *Given $(Q,[s])\in\mathcal{E}(G)$ and $(R,\psi)\in\mathcal{C}(G)$, then*

$$n_G(Q,s;R,\psi) = \sum_{z\in G,\, tQ\in[s]\,:\, Q\leq {}^zR,\, t\in N_G({}^zR)} {}^z\widehat{\psi}(t.{}^zR)\,|C_{\overline{N}_G(Q)}(s)|/|\overline{N}_G(Q)||N_G(R)|\ .$$

Proof. This follows from the previous proposition by substituting $z = g^{-1}$. The factor $1/|[s]| = |C_{\overline{N}(Q)}(s)|/|\overline{N}(Q)|$ appears because we are summing over all the elements of $[s]$. $\square$

Theorem 3.6. *Given $(P,\phi)\in\mathcal{C}(G)$ and $(Q,[s])\in\mathcal{E}(G)$, then*

$$n_G^{-1}(P,\phi;Q,s) = \sum_{vN_G(Q)\subseteq G,\, aP\in\overline{N}_G(P)_{p'}\,:\, P\leq {}^vQ,\, a\in N_G({}^vQ),\, a.{}^vQ\in {}^v[s]} \phi(a^{-1}P)\,\mu_a(P,{}^vQ)/|\overline{N}_G(P)|\ .$$

Proof. Note that the condition $P\leq {}^vQ$ implies that the conditions $a\in N({}^vQ)$ and $a.{}^vQ\in {}^v[s]$ depend only on aP, not on the choice of coset representative a. Let $\nu_G^{-1}(P,\phi;Q,s)$ denote the right-hand side of the asserted equality. Fixing (P,ϕ) and another element $(R,\psi)\in\mathcal{E}(G)$, let

$$\Delta = \sum_{(Q,[s])\in_G\mathcal{E}(G)} \nu_G^{-1}(P,\phi;Q,s)\,n_G(Q,s;R,\psi)\ .$$

If we can show that

$$\Delta = \begin{cases} 1 & \text{if } (P,\phi) =_G (R,\psi), \\ 0 & \text{otherwise,} \end{cases}$$

then the equality $n_G^{-1}(P,\phi;Q,s) = \nu_G^{-1}(P,\phi;Q,s)$ will follow. Summing over all the elements $(Q,[s])$ of $\mathcal{E}(G)$, we have

$$\Delta = \sum_{(Q,[s])\in\mathcal{E}(G)} \frac{|N(Q)|}{|G|}\,\nu_G^{-1}(P,\phi;Q,s)\,n_G(Q,s;R,\psi)\ .$$

Let $\Gamma = |\overline{N}(P)||N(R)|\Delta$. Applying Proposition 3.5, summing over all the elements v of G, we have

$$|G|\Gamma = \sum_{(Q,[s]),v,aP,z,tQ} \phi(a^{-1}P)\,\mu_a(P,{}^vQ).{}^z\widehat{\psi}(t.{}^zR)\,|C_{\overline{N}(Q)}(s)|/|\overline{N}(Q)|$$

with indices $(Q,[s])\in\mathcal{E}(G)$, $v\in G$, $aP\in\overline{N}(P)_{p'}$, $z\in G$, $tQ\in[s]$ subject to the conditions

$$P\leq {}^vQ\ ,\qquad a\in N({}^vQ)\ ,\qquad a.{}^vQ\in {}^v[s]\ ,\qquad Q\leq {}^zR\ ,\qquad t\in N({}^zR)\ .$$

Replacing tQ with $u.{}^vQ \in {}^v[s]$ where $u = {}^vt$, also replacing z with $y \in G$ where $y = vz$, we have ${}^z\widehat{\psi}(t.{}^zR) = {}^y\widehat{\psi}(u.{}^yR)$ and

$$|G|\Gamma = \sum_{(Q,[s]),v,aP,y,u.{}^vQ} \phi(a^{-1}P)\,\mu_a(P,{}^vQ).{}^y\widehat{\psi}(u.{}^yR)\,|C_{\overline{N}(Q)}(s)|/|\overline{N}(Q)|$$

where the indices are subject to

$$P \leq {}^vQ \leq {}^yR\,, \qquad a \in N({}^vQ)\,, \qquad a.{}^vQ \in {}^v[s]\,, \qquad u \in N({}^yR)\,.$$

As $(Q,[s])$ and v vary, the element $({}^vQ, {}^v[s])$ takes each value $|G|$ times. So

$$\Gamma = \sum_{(Q,[s]),aP,y,uQ} \phi(a^{-1}P)\,\mu_a(P,Q)\,.{}^y\widehat{\psi}(u.{}^yR)\,|C_{\overline{N}(Q)}(s)|/|\overline{N}(Q)|$$

with indices $(Q,[s]) \in \mathcal{E}(G)$, $aP \in \overline{N}(P)_{p'}$, $y \in G$, $uQ \in [s]$ subject to

$$P \leq Q \leq {}^yR\,, \qquad a \in N(Q)\,, \qquad aQ \in [s]\,, \qquad u \in N({}^yR)\,.$$

We now replace uQ with $cQ \in \overline{N}(Q)$ where $uQ = {}^{cQ}aQ$. For each value of uQ, there are $|C_{\overline{N}(Q)}(s)|$ associated values of cQ. We also replace y with $x \in G$ where $x = c^{-1}y$, obtaining

$$\Gamma = \sum_{(Q,[s]),aP,x,cQ} \phi(a^{-1}P)\,\mu_a(P,Q).{}^x\widehat{\psi}(a.{}^xR)/|\overline{N}(Q)|$$

with indices subject to

$$P \leq Q \leq {}^xR\,, \qquad a \in N(Q) \cap N({}^xR)\,, \qquad aQ \in [s]\,.$$

The term of the sum does not depend on cQ, so

$$\Gamma = \sum_{(Q,[s]),aP,x} \phi(a^{-1}P)\,\mu_a(P,Q).{}^x\widehat{\psi}(a.{}^xR)\,.$$

The conditions $aP \in \overline{N}(P)_{p'}$ and $a \in N(Q) \cap N({}^xR)$ imply that aQ is a p'-element of $\overline{N}(Q)$. So we can remove $[s]$ from the indexing, obtaining

$$\Gamma = \sum_{x,aP} \phi(a^{-1}P).{}^x\widehat{\psi}(a.{}^xR) \sum_Q \mu_a(P,Q)$$

with the indices $x \in G$ and $aP \in \overline{N}(P)_{p'}$ satisfying $P \leq {}^xR$ and $a \in N({}^xR)$, the index Q running over the subgroups $P \leq Q \leq {}^xR$ such that $a \in N(Q)$. In other words, Q runs over the elements of the closed interval $[P, {}^xR]$ in the poset $\mathcal{S}_p(G)^{\langle a \rangle}$. We now apply the recurrence relation for Möbius functions. If $P \neq_G R$ then, for each x and aP, we have $\sum_Q \mu_a(P,Q) = 0$, hence $\Gamma = 0$ and $\Delta = 0$.

It remains only to deal with the case where $P =_G R$. We may assume that $P = R$ and it suffices to show that if $\phi = \psi$ then $\Delta = 1$, otherwise $\Delta = 0$. We have $\sum_Q \mu_a(P,Q) = \mu_a(P,P) = 1$. Using the latest formula for Γ and the orthogonality relation for Brauer characters,

$$\Delta = \sum_{x \in N(P), aP \in \overline{N}(P)_{p'}} \phi(a^{-1}P).{}^x\widehat{\psi}(a.{}^xP)/|\overline{N}(P)||N(P)|$$

$$= \sum_{aP \in \overline{N}(P)_{p'}} \phi(a^{-1}P).\widehat{\psi}(aP)/|\overline{N}(P)| = \begin{cases} 1 & \text{if } \phi = \psi, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

The formula in the theorem still holds if we divide the term by $|N(Q)|$ and sum over all the elements $v \in G$ instead of the coset representatives $vN(Q) \subseteq G$. Now making the substitutions $w = v^{-1}$ and $g = {}^w a$, the formula becomes

$$n_G^{-1}(P, \phi; Q, s) = \sum_{w \in G,\, g.{}^wP \in \overline{N}_G({}^wP)_{p'} \,:\, {}^wP \leq Q,\, b \in N_G(Q),\, gQ \in [s]} {}^w\phi(g^{-1}.{}^wP)\, \mu_g({}^wP, Q)/|\overline{N}_G(P)||N_G(Q)| \,.$$

Meanwhile, summing over all the elements (P, ϕ) of $\mathcal{C}(G)$, we have

$$e_{Q,s}^G = \frac{1}{|G|} \sum_{(P,\phi) \in_G \mathcal{C}(G)} |N(P)|\, n_G^{-1}(P, \phi; Q, s)\, [N_{P,\phi}^G] \,.$$

Combining the latest two equalities, and noting that each element of $\mathcal{C}(G)$ appears $|G|$ times as $({}^wP, {}^w\phi)$, we obtain the formula for $e_{Q,s}^G$ in Theorem 2.4.

4 The linearization map

Letting $V \leq G$ and letting ν be the Brauer character of a 1-dimensional $\mathbb{F}V$-module $\mathbb{F}_\nu$, then ${}_G\mathrm{Ind}_V(\mathbb{F}_\nu)$ is a trivial source $\mathbb{F}G$-module. In this section, we shall explicitly express the isomorphism class $[{}_G\mathrm{Ind}_V(\mathbb{F}_\nu)]$ as a linear combination of the elements of the canonical basis of $T_\mathbb{F}(G)$.

Let us make some comments on interpretation. An **$\mathbb{F}^\times$-fibred G-set**, recall, is defined to be an $\mathbb{F}^\times$-free $\mathbb{F}^\times \times G$-set with finitely many $\mathbb{F}^\times$-orbits. The $\mathbb{F}^\times$-orbits are called the **fibres**. The **$\mathbb{F}$-monomial Burnside ring**, denoted $\mathbb{B}_\mathbb{F}(G)$, is defined to be the $\mathbb{Z}$-module generated by the isomorphism classes of $\mathbb{F}^\times$-fibred G-sets, subject to the relation $[X] + [Y] = [X \sqcup Y]$ where X and Y are $\mathbb{F}^\times$-fibred G-sets and $[X]$ denotes the isomorphism class of X. We make $B_\mathbb{F}(G)$ become a unital ring with multiplication given by $[X][Y] = [X \otimes Y]$ where $X \otimes Y$ is the set of $F^\times$-orbits of $X \times Y$ under the action such that $\lambda \in \mathbb{F}^\times$ sends $(x, y) \in X \times Y$ to $(\lambda x, \lambda^{-1} y)$. Let $\mathbb{F}X$ denote the $\mathbb{F}G$-module, well-defined up to isomorphism, such that there is an embedding of $\mathbb{F}^\times \times G$-sets $X \hookrightarrow \mathbb{F}X$ whereby any set of representatives of the fibres of X becomes a basis for $\mathbb{F}X$. Let

$$\mathrm{lin}_G \,:\, B_\mathbb{F}(G) \to T_\mathbb{F}(G)$$

be the unital ring homomorphism such that $\mathrm{lin}_G[X] = [\mathbb{F}X]$. We mention that lin_G is surjective and that $\mathrm{lin} : B_\mathbb{F} \to T_\mathbb{F}$ is a morphism of biset functors. For those two results and further discussions of $B_\mathbb{F}(G)$ and the biset functor $B_\mathbb{F}$, see [Bar04] and Boltje [Bol98], [Bol]. The isomorphism classes of transitive $F^\times$-fibred G-sets comprise a $\mathbb{Z}$-basis for $B_\mathbb{F}(G)$. These basis elements have the form $[{}_G\mathrm{Ind}_V(\mathbb{F}_\nu^\times)]$ where V and ν are as above and $\mathbb{F}_\nu^\times = \mathbb{F}_\nu - \{0\}$. We have $\mathrm{lin}_V[\mathbb{F}_\nu^\times] = [\mathbb{F}_\nu]$ and

$$\mathrm{lin}_G[{}_G\mathrm{Ind}_V(\mathbb{F}_\nu^\times)] = [{}_G\mathrm{Ind}_V(\mathbb{F}_\nu)] \,.$$

The formula in the next theorem can be viewed as a formula for the matrix of lin_G with respect to the above basis for $B_\mathbb{F}(G)$ and the canonical basis for $T_\mathbb{F}(G)$.

For a p-subgroup P of G and an arbitrary subgroup V of G, we write $(P, V]_{\mathcal{S}_p}$ to denote the poset of p-subgroups Q such that $P < Q \leq V$. Given $g \in N_G(P) \cap V$, we write $(P, V]_{\mathcal{S}_p}^{\langle g \rangle}$ to denote the subposet of $(P, V]_{\mathcal{S}_p}$ consisting of those Q such that $g \in N_G(Q)$.

Theorem 4.1. *Given V and ν as above, then*

$$[{}_G\mathrm{Ind}_V(\mathbb{F}_\nu)] = \frac{-1}{|V|} \sum_{(P,\phi)\in\mathcal{C}(G),\, gP\in\overline{N}_G(P)_{p'}\,:\,\langle P,g\rangle\le V} |P|\,\phi(g^{-1}P)\,\nu(g)\,\widetilde{\chi}(([P,V]^{\langle g\rangle}_{\mathcal{S}_p})\,[N^G_{P,\phi}]\;.$$

Proof. Let $I = [{}_G\mathrm{Ind}_V(F_\nu)]$. Summing over representatives of G-orbits, write

$$I = \sum_{(P,\phi)\in_G\mathcal{C}(G)} \lambda_{P,\phi}[N^G_{P,\phi}]\;.$$

We are to evaluate the coefficients $\lambda_{P,\phi}\in\mathbb{K}$. Consider an element $(Q,[s])\in\mathcal{E}(G)$ and let $\widehat{s}\in N_G(Q)$ such that $s=\widehat{s}Q$. We have

$$({}_G\mathrm{Ind}_V(\mathbb{F}_\nu))[Q] = \bigoplus_{fV\subseteq G\,:\,Q\le {}^fV} f\otimes\mathbb{F}_\nu\;.$$

Supposing that $Q\le {}^fV$, then s stabilizes $f\otimes\mathbb{F}_\nu$ if and only if $\langle Q,\widehat{s}\rangle\le {}^fV$, in which case, s acts on $f\otimes\mathbb{F}_\nu$ as multiplication by ${}^f\nu(\widehat{s})$. Therefore

$$\epsilon^G_{Q,s}(I) = \sum_{fV\subseteq G\,:\,\langle Q,\widehat{s}\rangle\le {}^fV} {}^f\nu(\widehat{s})\;.$$

It follows that

$$I = \sum_{(Q,[s])\in_G\mathcal{E}(G)} \epsilon^G_{Q,s}(I)\,e^G_{Q,s} = \frac{1}{|G|}\sum_{(Q,[s])\in\mathcal{E}(G),fV\subseteq G\,:\,\langle Q,\widehat{s}\rangle\le {}^fV} |N_G(Q)|.{}^f\nu(\widehat{s})\,e^G_{Q,s}\;.$$

Note that there is an arbitrary choice of representative element s of $[s]$ and there is an arbitrary choice of lift $\widehat{s}$ of s. Given $(Q,[s])$, then the range of the index fV depends on the choice of s but, given $(Q,[s])$ and s, then the range of fV does not depend on the choice of $\widehat{s}$. Applying Theorem 2.4,

$$I = \frac{1}{|G|}\sum_{(Q,[s]),fV,(P,V),gP} |P|.{}^f\nu(\widehat{s})\,\phi(g^{-1}P)\,\mu_g(P,Q)\,[N^G_{P,\phi}]$$

summed over $(Q,[s])\in\mathcal{E}(G)$, $fV\subseteq G$, $(P,\phi)\in\mathcal{C}(G)$, $gP\in\overline{N}_G(P)_{p'}$ subject to

$$\langle Q,\widehat{s}\rangle\le {}^fV\;,\qquad P\le Q\;,\qquad g\in N_G(Q)\;,\qquad gQ\in[s]\;.$$

As we noted above, given $(Q,[s])$, the choices of s and $\widehat{s}$ are arbitrary. So, changing the order of summation by giving the index gP priority over the index fV, we may replace $\widehat{s}$ with g. Bearing in mind that (P,ϕ) has $|G:N_G(P)|$ conjugates, we obtain

$$\lambda_{P,\phi} = \frac{1}{|N_G(P)|}\sum_{gP,fV} |P|.{}^f\nu(g)\,\phi(g^{-1}P)\sum_Q \mu_g(P,Q)$$

summed over $gP\in\overline{N}_G(P)_{p'}$, $fV\subseteq G$, $Q\in\mathcal{S}_p(G)$ subject to the two conditions

$$P\le Q\le {}^fV\;,\qquad g\in N_G(Q)\cap {}^fV\;.$$

Those two conditions can be rewritten as

$$\langle P,g\rangle\le {}^fV\;,\qquad Q\in[P,{}^fV]^{\langle g\rangle}_{\mathcal{S}_p}$$

where $[P, {}^fV]^{\langle g\rangle}_{\mathcal{S}_p}$ denotes the set of p-subgroups Q of G such that $P \leq Q \leq {}^fV$ and $g \in N_G(P)$. By the definition of the Möbius function,

$$\mu_g(P, Q) = \sum_{R_{-1},...,R_{n+1}} (-1)^n$$

summed over $R_{-1}, ..., R_{n+1} \in [P, {}^fV]^{\langle g\rangle}_{\mathcal{S}_p}$ such that $P = R_{-1} < ... < R_{n+1} = Q$ (allowing the possibilities $n = -2$ and $n = -1$). Therefore,

$$\sum_Q \mu_g(P, Q) = -\widetilde{\chi}((P, {}^fV]^{\langle g\rangle}_{\mathcal{S}_p}) \; .$$

We have shown that

$$\lambda_{P,\phi} = \frac{-1}{|N_G(P)|} \sum_{gP \in \overline{N}_G(P)_{p'}, fV \subseteq G \,:\, \langle P, g\rangle \leq {}^fV} |P| \, \phi(G^{-1}P).{}^f\nu(g) \, \widetilde{\chi}((P, {}^fV]^{\langle g\rangle}_{\mathcal{S}_p}) \; .$$

Again bearing in mind that (P, ϕ) has $|G : N_G(P)|$ conjugates,

$$I = \frac{-1}{|G|} \sum_{(P,\phi), gP, fV} |P| \, \phi(G^{-1}P).{}^f\nu(g) \, \widetilde{\chi}((P, {}^fV]^{\langle g\rangle}_{\mathcal{S}_p}) \, [N^G_{P,\phi}]$$

summed over $(P, \phi) \in \mathcal{C}(G)$ and indices gP and fV as before. Dividing by $|V|$ and replacing the index fV with $f \in G$, then making the substitution $h = f^{-1}$, we have

$$I = \frac{1}{|G||V|} \sum_{(P,\phi), gP, h} |P|.{}^h\phi({}^h(g^{-1}P)) \, \nu({}^hg) \, \widetilde{\chi}(({}^hP, V]^{\langle {}^hg\rangle}_{\mathcal{S}_p}) \, [N^G_{{}^hP, {}^h\phi}]$$

summed over $(P, \phi) \in \mathcal{C}(G)$, $gP \in \overline{N}_G(P)_{p'}$, $h \in G$ such that $\langle {}^hP, {}^h g\rangle \leq G$. Rearranging the sum so that h becomes the first index, then replacing P, ϕ, g with P^h, ϕ^h, g^h, we obtain the required equality. □

Corollary 4.2. *Given V and ν as above, writing $[{}_G\mathrm{Ind}_V(F_\nu)]$ as a linear combination of the elements of the canonical basis for $T_{\mathbb{F}}(G)$ and supposing that a given basis element $[N^G_{P,\phi}]$ has non-zero coefficient, then $O_p(V) \leq {}^xP \leq V$.*

Proof. By the latest theorem, there exist $x \in G$ and $g \in N_G({}^xP)$ such that $\langle {}^xP, g\rangle \leq V$ and $\widetilde{\chi}(({}^xP, V]^{\langle g\rangle}_{\mathcal{S}_p}) \neq 0$. For a contradiction, suppose that $O_p(V) \not\leq {}^xP$. We may assume that $x = 1$. Then the poset $(P, V]^{\langle g\rangle}_{\mathcal{S}_p}$ admits a conical contraction $Q \mapsto QO_p(V) \mapsto PO_p(V)$, contradicting the condition that the reduced Euler characteristic is non-zero. □

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