MATH 524: Algebra II.

Midterm Practise Exercises.

LJB, 10 March 2015, Bilkent University.

1: Let α_1, ..., α_n be algebraic numbers. Let $E = \mathbb{Q}[\alpha_1, ..., \alpha_n]$.
(a) State and prove an inequality relating $[E : \mathbb{Q}]$ to the degrees $[\mathbb{Q}[\alpha_j] : \mathbb{Q}]$.
(b) Suppose that $n = 4$, and that the minimal polynomials of α_1, ..., α_4 are $X^2 - 2$, $X^3 - 3$, $X^5 - 5$, $X^7 - 7$. Evaluate $[E : \mathbb{Q}]$.

2: Let p be a prime and let $1 \leq m \leq n$ be integers. State and prove a necessary and sufficient condition on m and n for $\mathbb{F}_{p^n}$ to be an extension of the field $\mathbb{F}_{p^m}$.

3: Let $f_1(X)$, ..., $f_n(X)$ be polynomials over $\mathbb{Q}$ with degree 2. Let L be the splitting field for $f_1(X)...f_n(X)$ over $\mathbb{Q}$.
(a) Show that the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is abelian. (Hint: consider the squares of the group elements.)
(b) Now suppose that the degree of L over $\mathbb{Q}$ is $|L : \mathbb{Q}| = 8$. How many fields K are there such that $\mathbb{Q} \leq K \leq L$?

4: Let F be a field with characteristic zero, let $f(X)$ be an irreducible quartic (degree 4) polynomial over F, let E be the splitting field for $f(X)$ over F, and let G be the Galois group of the extension E/F.
(a) State the Fundamental Theorem of Galois Theory, including clauses concerning the order of the Galois group and the normal subgroups of the Galois group.
(b) Let α_1, α_2, α_3, α_4 be the roots to $f(X)$, and let

$$\delta = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4)(\alpha_3 - \alpha_4)\,.$$

Suppose that $\delta \in F$. Show that $G \cong V_4$ or $G \cong A_4$.
(c) In each of the two cases in part (b), find the number of intermediate fields $F \leq L \leq E$. In each of those two cases, how many of those L are normal extensions of F?

5: Let E be the splitting field for $X^6 - 2$ over $\mathbb{Q}$.
(a) Show that the Galois group for E over $\mathbb{Q}$ is the dihedral group with order 12.
(c) How many intermediate fields $\mathbb{Q} \leq K \leq E$ are there?