How phases appear when reducing to quotient groups, for instance, as in Alperin's Conjecture

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Abstract: This talk is a pedagogical introduction. Clifford theory reduces to quotient groups, but with the introduction of a twist: a phase factor, a cohomology class. Alperin's Conjecture says that the number of simples of a block is "locally determined" in terms of subquotients $N(P)/P$ for p-subgroups P. After Puig, "locally determined" means determined by the fusion system up to finite information. But the fusion system, a category consisting of conjugation homomorphisms between p-subgroups, cannot see the subquotients $PC(P)/P$. To express Alperin's Conjecture in a truly local way, Clifford theory is applied to quotient out $PC(P)/P$ from $N(P,e)/P$, where e is a suitable block of $PC(P)$. The extra information in the introduced twists is finite.

1: Twisted group algebras

Clifford theory (named after Alfred Clifford, 20th century, not William Kingdon Clifford, 19th) is a technique for replacing an arbitrary normal subgroup with a central cyclic subgroup whose elements can be identified with units in a field. We shall be discussing Clifford theory of blocks in the context of Alperin's Conjecture. Further details can be found in Kessar's introductory review [Kes07]. (Thus, this talk is an introduction to an introduction.) For a historical account and a discussion of outlook, see Puig [Pui09].

A **twisted group** over a field $\mathbb{F}$ is defined to be a pair (α, G), also written as ${}_\alpha G$, where G is a group and α is a function from $G \times G$ to the unit group $\mathbb{F}^\times = \mathbb{F} - \{1\}$ satisfying

$$\alpha(f,g)\,\alpha(fg,h) = \alpha(f,gh)\,\alpha(g,h)$$

for all $f, g, h \in G$. Two twisted groups ${}_{\alpha'}G'$ and ${}_\alpha G$ are understood to be the same provided $G' = G$ and $\alpha' \equiv \alpha$, we mean, there exists a function $\beta : G \to \mathbb{F}^\times$ satisfying

$$\alpha'(g,h)/\alpha(g,h) = \beta(g)\beta(h)/\beta(gh)$$

for all g, h. An element of $\mathbb{F}^\times$, in this context, is called a **phase**. The function α is called a **cocycle**. The equivalence class $[\alpha]$ is called a (second degree) **cohomology class** or, more casually, a **twist**. The twists, equipped with a suitable binary operation, comprise an abelian group $H^2(G, \mathbb{F}^\times)$ called the (second degree) **cohomology group** of G over $\mathbb{F}^\times$.

We define the **twisted group algebra** of ${}_\alpha G$, denoted $\mathbb{F}_\alpha G$, to be the algebra over $\mathbb{F}$ with basis $\{{}_\alpha g : g \in G\}$ and multiplication given by

$${}_\alpha g\, {}_\alpha h = \alpha(g,h)\, {}_\alpha(gh)\ .$$

The above relation says precisely that the multiplication is associative, $({}_\alpha f\, {}_\alpha g)_\alpha h = {}_\alpha f({}_\alpha g\, {}_\alpha h)$. When $\alpha' = \alpha$, we can identify $\mathbb{F}_{\alpha'}G = \mathbb{F}_\alpha G$ by understanding that ${}_{\alpha'}g = \beta(g)\, {}_\alpha g$ with β as above. Thus, fixing G, the elements $[\alpha] \in H^2(G, \mathbb{F}^\times)$ correspond bijectively with the twisted groups ${}_\alpha G$ and also with the twisted group algebras $\mathbb{F}_\alpha G$.

For example, the algebra of quaternions $\mathbb{H} = \mathrm{span}_\mathbb{R}\{1, i, j, k : i^2 = j^2 = k^2 = ijk = -1\}$ becomes a twisted group algebra over the group $V_4 = \{1, x, y, z : x^2 = y^2 = z^2 = xyz = 1\}$

via the identifications $i = {}_\alpha x$ and $j = {}_\alpha y$ and $k = {}_\alpha z$ where α is as shown in the table, -1 abbreviated as $-$. Thus, $\mathbb{H} \cong \mathbb{R}_\alpha V_4$ and $\mathbb{R}Q_8 \cong \mathbb{R}^4 \oplus \mathbb{H} \cong \mathbb{R}V_4 \oplus \mathbb{R}_\alpha V_4$.

$\alpha(g,h)$	1	x	y	z	h
1	1	1	1	1	
x	1	$-$	1	$-$	
y	1	$-$	$-$	1	
z	1	1	$-$	$-$	
g					

It is perhaps worth pointing out that the quaternions first arose, though the work of Rodrigues and Hamilton, as a calculus for 3-dimensional rotations. But, for $q \in \mathbb{H}$ with $|q| = 1$, the rotations associated with q and $-q$ coincide. In particular, Q_8 acts as the rotational symmetry group V_4 of a rectangular cuboid, say, where $\pm i$ acts as x, also $\pm j$ acts as y, also $\pm k$ acts as z. In a sense that has considerable significance in quantum electrodynamics, the phase factors ± 1 do not have any effect on classical observables.

2: Alperin's Conjecture

A primitive idempotent of $Z(\mathbb{F}G)$ is called a **block** of $\mathbb{F}G$. Plainly, every idempotent of $Z(\mathbb{F}G)$ is a sum of blocks, the sum of all the blocks is $1 = \sum_b b$, and

$$\mathbb{F}G = \bigoplus_b \mathbb{F}Gb\ .$$

as a direct sum of indecomposable algebras. Any indecomposable $\mathbb{F}G$-module, perforce any simple $\mathbb{F}G$-module, is an $\mathbb{F}Gb$ module for a unique block b.

The following two facts are special cases of properties of finite-dimensional algebras over fields: the indecomposable projective $\mathbb{F}G$-modules are precisely the indecomposable modules of the regular $\mathbb{F}G$-module ${}_{\mathbb{F}G}\mathbb{F}G$. The condition that S is the unique simple quotient of T characterizes a bijective correspondence $S \leftrightarrow T$ between the isomorphism classes of simple $\mathbb{F}G$-modules S and the isomorphism classes of indecomposable projective $\mathbb{F}G$-modules T.

It can be shown that there exists a simple projective $\mathbb{F}Gb$-module if and only if $\mathbb{F}Gb$ is a full matrix ring $\mathrm{Mat}_n(\Delta)$ over a division ring Δ. In that case, the simple projective module $T \cong S$ is, up to isomorphism, the unique indecomposable $\mathbb{F}Gb$-module. Then we call b a **defect-zero block**. Otherwise, we call b a **positive-defect block**. Maschke's Theorem says that every block of $\mathbb{F}G$ is a defect-zero block if and only if the characteristic of $\mathbb{F}$ does not divide $|G|$.

Henceforth, let us assume that $\mathbb{F}$ is algebraically closed and with prime characteristic p. The following two natural numbers depend only on p and G:

• the number $z(\mathbb{F}G)$ of defect-zero blocks of $\mathbb{F}G$, equal to the number of isomorphism classes of simple projective $\mathbb{F}G$-modules, also equal to the number of irreducible $\mathbb{C}G$-characters χ such that the integer $|G|/\chi(1)$ is divisible by p.
• the number $\ell(\mathbb{F}G)$ of isomorphism classes of simple $\mathbb{F}G$-modules, equal to the number of isomorphism classes of indecomposable projective $\mathbb{F}G$-modules, also equal to the number of p-regular conjugacy classes of G.

The weak form of Alperin's conjecture is deceptive; it looks like a recursive formula for $z(\mathbb{F}G)$, which is harder to determine than $\ell(\mathbb{F}G)$. It says that, letting P run over representatives of the conjugacy classes of p-subgroups of G, and writing $\overline{N}_G(P) = N_G(P)/P$, then

$$\ell(\mathbb{F}G) = \sum_P z(\mathbb{F}\overline{N}_G(P)) \ .$$

But Alperin conjectured, more strongly, that for any idempotent e of $Z(\mathbb{F}G)$, we have

$$\ell(\mathbb{F}Ge) = \sum_P z(\mathbb{F}\overline{N}_G(P)\overline{\mathrm{br}_P(e)})$$

where ℓ and z are generalized in an evident way and the idempotent $\overline{\mathrm{br}_P(e)} \in Z(\mathbb{F}\overline{N}_G(P))$ is the modulo P reduction of an idempotent $\mathrm{br}_P(e) \in Z(\mathbb{F}N_G(P))$. Now replacing e with a positive-defect block b, the summand for $P = 1$ is $z(\mathbb{F}Gb) = 0$ and the conjectural formula makes sense as a means of evaluating the $\ell(\mathbb{F}Gb)$ in terms of z for strictly smaller groups.

3: Fusion systems

The formula can be modified by introducing pairs (P, ϵ), called **Brauer pairs**, where P is a p-subgroup of G and ϵ is a block of $\mathbb{F}PC_G(P)$. A lucid account of their fundamental properties can be found in Alperin–Broué [AB79], where they are called *subpairs*, the name reflecting the idea that they are to be regarded as generalizations of p-subgroups. The Brauer pairs for $\mathbb{F}G$ admit a partial ordering $\leq$ which we shall not define. The only property we need to state is that, given another p-subgroup P', then $P' \leq P$ if and only if there exists an ϵ' satisfying $(P', \epsilon') \leq (P, \epsilon)$, in which case, ϵ' is unique. See [AB79, 3.4].

We choose and fix a maximal Brauer pair (D, e_D) such that $(1, b) \leq (D, e_D)$. The property stated above implies that, for each $P \leq D$, there exists a unique block e_P of $\mathbb{F}PC_G(P)$ such that $(P, e_P) \leq (D, e_D)$. The same property also implies that $b = e_1$ and that b is the unique block of $\mathbb{F}G$ such that $(1, b) \leq (P, e_P)$.

The set of all Brauer pairs for $\mathbb{F}G$ admits an evident conjugation action of G. The Brauer subpairs (P, e_P) of (D, e_D) are not necessarily stabilized under that action. But we can define a category $\mathcal{F} = \mathcal{F}(\mathbb{F}Gb)$, called the **fusion system** of $\mathbb{F}Gb$, whose objects are the subgroups of D, the morphisms $P \leftarrow Q$ being the conjugation homomorphisms ${}^g x \leftarrow\!\shortmid x$ coming from those elements $g \in G$ such that $(P, e_P) \geq {}^g(Q, e_Q)$. A version of Sylow's Theorem for Brauer pairs, in [AB79, 3.10], tells us that, given b, then the maximal Brauer pair (D, e_D) is well-defined up to G-conjugation, hence the category $\mathcal{F}$ is well-defined up to isomorphism.

It is not hard to reformulate Alperin's conjecture as

$$\ell(\mathbb{F}Gb) = \sum_{P \leq_{\mathcal{F}} D} z(\mathbb{F}\overline{N}_G(P, e_P)\overline{e_P})$$

where $\overline{N}_G(P, e_P) = N_G(P, e_P)/P$ and P now runs over representatives of the isomorphism classes in $\mathcal{F}$. The passage from $\overline{N}_G(P)$ to $\overline{N}_G(P)$ is done by a preliminary little step in the Clifford theoretic technique. We shall comment on it further in the next section.

As an example, putting $p = 3$ and $G = \mathrm{GL}_2(3)$, the regular $\mathbb{F}G$-module decomposes as

$${}_{\mathbb{F}G}\mathbb{F}G \cong \begin{pmatrix} 1_1 \\ 1_2 \\ 1_1 \end{pmatrix} \oplus \begin{pmatrix} 1_2 \\ 1_1 \\ 1_2 \end{pmatrix} \oplus 3(3_1) \oplus 3(3_2) \oplus 2 \begin{pmatrix} 2_1 \\ 2_2 \\ 2_1 \end{pmatrix} \oplus 2 \begin{pmatrix} 2_2 \\ 2_1 \\ 2_2 \end{pmatrix} .$$

The notation, here, indicates that all the summands are uniserial modules, expressions n_j denoting an n-dimensional simple module. The first two of the six terms are inflated from

$S_3 = \mathrm{GL}_2(3)/Q_8$, and the first four terms are inflated from $S_4 = \mathrm{GL}_2(3)/\langle z \rangle$ where z is the central involution. We mean to say, the regular modules for $\mathbb{F}S_3$ and $\mathbb{F}S_4$ are

$$ {}_{\mathbb{F}S_3}\mathbb{F}S_3 \cong \begin{pmatrix} 1_1 \\ 1_2 \\ 1_1 \end{pmatrix} \oplus \begin{pmatrix} 1_2 \\ 1_1 \\ 1_2 \end{pmatrix} , \qquad {}_{\mathbb{F}S_4}\mathbb{F}S_4 \cong \begin{pmatrix} 1_1 \\ 1_2 \\ 1_1 \end{pmatrix} \oplus \begin{pmatrix} 1_2 \\ 1_1 \\ 1_2 \end{pmatrix} \oplus 3(3_1) \oplus 3(3_2) . $$

The weak form of Alperin's Conjecture, in this case, says that

$$ 6 = \ell(\mathbb{F}G) = z(\mathbb{F}N_G(1)) + z(\mathbb{F}N_G(C_3)) = z(\mathbb{F}G) + z(\mathbb{F}V_4) = 2 + 4 . $$

Putting $b = (1 - z)/2 = z - 1$, the algebra $\mathbb{F}Gb$ is a twisted group algebra over S_4, and its regular module is

$$ {}_{\mathbb{F}Gb}\mathbb{F}Gb \cong 2 \begin{pmatrix} 2_1 \\ 2_2 \\ 2_1 \end{pmatrix} \oplus 2 \begin{pmatrix} 2_2 \\ 2_1 \\ 2_2 \end{pmatrix} . $$

As we shall see in the next section, it is no coincidence that the structure looks very similar to that for $\mathbb{F}S_3$. It turns out that (C_3, b) is a maximal Brauer pair containing $(1, b)$. Alperin's Conjecture for b says that

$$ 2 = \ell(\mathbb{F}Gb) = z(\mathbb{F}\overline{N}_G(C_3)\overline{b}) = z(\mathbb{F}V_4\overline{b}) = 2 . $$

Perhaps one might feel that the above formula for $\ell(\mathbb{F}Gb)$ is a satisfactory rendition of Alperin's Conjecture in terms involving the fusion system $\mathcal{F}$. But one of Puig's guiding principles was that $\mathcal{F}$ should determine the essence of the block algebra $\mathbb{F}Gb$ up to a finite amount of information. The trouble is that if we pass from the case $G = \mathrm{GL}_2(3)$ to the case $G = \mathrm{GL}_2(3) \times G'$ where G' is of order one-zillion (not divisible by 3), taking b much as before, with G' acting trivially on the $\mathbb{F}Gb$-modules, then $\mathcal{F}$ does not change, the above calculation $2 = 2$ does not change, but the group $\overline{N}_G(C_3, b)$ increases in size by a factor of one-zillion. From $\mathcal{F}$, one can recover the group

$$ \underline{\overline{N}}_G(P, e_P) = N_G(P, e_P)/PC_G(P) \cong \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P) $$

but one cannot recover the group $\overline{N}_G(P, e_P) = N_G(P, e)/P$ because of the infinite amount of spurious variability of the normal subgroup

$$ \overline{C}_G(P) = PC_G(P)/P \cong C_G(P)/Z(P) . $$

We use Clifford theory to get rid of that normal subgroup. The subgroup $P \leq D$ is said to be **$\mathcal{F}$-centric** provided $\overline{e_P}$ is a defect-zero block of $\mathbb{F}\overline{C}_G(P)$. It can be shown that this condition is necessary for $z(\mathbb{F}\overline{N}_G(P, e)\overline{e_P})$ to be non-zero. It can also be shown that this condition can be detected using only the data encoded into the fusion system. For such P, the algebra $\mathbb{F}\overline{N}_G(P, e_P)\overline{e_P}$ is equivalent to a twisted group algebra $\mathbb{F}_{\alpha_P}\underline{N}_G(P, e_P)$ in a sense that will be explained at the beginning of the next section. In particular,

$$ \ell(\mathbb{F}Ge) = \sum_P z(\mathbb{F}_{\alpha_P}\underline{N}_G(P, e_P)) $$

now summed over the $\mathcal{F}$-isomorphism classes of $\mathcal{F}$-centric subgroups P. Thus, $\ell(\mathbb{F}Ge)$ is determined by $\mathcal{F}$ together with the twists α_P.

For the above example with $G = \mathrm{GL}_2(3)$ or $G = \mathrm{GL}_2(3) \times G'$, the twist is trivial and

$$2 = \ell(\mathbb{F}Gb) = z(\mathbb{F}C_2) = 2\ .$$

4: Clifford theory of defect-zero blocks

We shall be reducing some finite-dimensional algebras over $\mathbb{F}$ to some smaller but equivalent algebras. As well as indicating the main ideas in behind the above reformulations of Alperin's Conjecture, one of our tasks will be to supply a rationale as to why the twisted group algebra of S_4 that appeared in the previous section has indecomposable projective modules that look similar to those of the ordinary group algebra of S_3.

Let us begin by discussing the appropriate notion of equivalence. When we speak of algebras over $\mathbb{F}$, we mean finite-dimensional algebra over $\mathbb{F}$. Given an algebra A over $\mathbb{F}$, when we speak of an A-module, we mean a finite-dimensional A-module. The category of A-modules, denoted A–Mod, is $\mathbb{F}$-linear (sometime called $\mathbb{F}$-preadditive): the morphism sets are $\mathbb{F}$-vector spaces and the composition operation is $\mathbb{F}$-bilinear. In the study of $\mathbb{F}$-linear categories, all functors and natural transformations are understood to be $\mathbb{F}$-linear in evident senses.

Two algebras A and B over $\mathbb{F}$ are said to be **Morita equivalent** provided A–Mod and B–Mod are equivalent as $\mathbb{F}$-linear categories. The algebras Morita equivalent to A are precisely those that are isomorphic to $\mathrm{End}_A(m_1T_1 \oplus ... \oplus m_kT_k)^\circ$ where T_1, ..., T_k are representatives of the isomorphism classes of indecomposable projective A-modules, m_1, ..., m_k are positive integers and the superscript $\circ$ indicates the opposite algebra. In particular, A is Morita equivalent to the matrix algebra $\mathrm{Mat}_n(A) \cong A \otimes_{\mathbb{F}} \mathrm{Mat}_n(\mathbb{F})$ for any positive integer n.

Versions of the following theorem, in various contexts, are often used as a preliminary reduction in applications of Clifford theory. They all reduce to the case where something associated with a normal subgroup is stable under conjugation.

Theorem 4.1: (Fong–Reynolds Theorem.) *Let $K \trianglelefteq G$ and let e be a defect-zero block of $\mathbb{F}K$. Write $N_G(e)$ for the stabilizer of e under the conjugation action of G on $\mathbb{F}K$. Write ϵ for the sum of the G-conjugates of e. Then induction ${}_G\mathrm{Ind}_{N_G(e)} : \mathbb{F}G\epsilon\text{–Mod} \leftarrow \mathbb{F}N_G(e)e\text{–Mod}$ yields a Morita equivalence between $\mathbb{F}G\epsilon$ and $\mathbb{F}N_G(e)$.*

The theorem lies behind the replacement of $\overline{N}_G(P)$ with $\overline{N}(P, e_P)$ in the previous section.

To illustrate the use of the theorem, we give an example which is not in the form of a reduction from a group $\overline{N}_G(P)$ to a group $\overline{N}_G(P, e_P)$, but which has the advantage of being small and very easy. Suppose that $\mathrm{char}(\mathbb{F}) \neq 2$ and $G = A_4$ and $K = V_4$ and e is one of the three mutually G-conjugate blocks of $\mathbb{F}K$. We immediately deduce that, letting S be the isomorphically unique 1-dimensional $\mathbb{F}Ke$-module, then the 3-dimensional $\mathbb{F}G$-module ${}_G\mathrm{Ind}_K(S)$ is simple.

For the rest of this talk, we assume that the defect-zero block e of $\mathbb{F}K$ is G-stable. Then e is an idempotent of $Z(\mathbb{F}G)$. We shall explain how the algebra $\mathbb{F}Ge$ is Morita equivalent to a twisted group algebra of the group $\overline{G} = G/K$.

The Noether–Skolem Theorem asserts that every automorphism of a matrix ring over a division ring is the conjugation action of an invertible matrix. We have $\mathbb{F}Ke \cong \mathrm{Mat}_n(\mathbb{F})$ for some n. So, for each $g \in G$, we can choose a unit $\widehat{g} \in \mathbb{F}Ke$ such that g and $\widehat{g}$ have the same conjugation action on $\mathbb{F}Ke$. We insist that the choices be made such that $\widehat{gk} = \widehat{g}\,k$ for all $g \in G$ and $k \in K$. That condition can be ensured by first letting g run over representatives of the non-trivial cosets of K in H, choosing $\widehat{g}$ arbitrarily for each of those g, whereupon $\widehat{g}$ is determined for all the other elements g of G.

Lemma 4.2 *For all $g \in G$ and $k \in K$, we have $\widehat{gk} = \widehat{g}\,\widehat{k}$ and $\widehat{kg} = \widehat{k}\,\widehat{g}$.*

Proof: We have $\widehat{gk} = \widehat{g}\,k = \widehat{g}\,\widehat{k}$ and $\widehat{kg} = \widehat{gk^g} = \widehat{g}\,k^g = \widehat{g}\,k^g\,\widehat{g}^{-1}\,\widehat{g} = k\,\widehat{g} = \widehat{k}\,\widehat{g}$. $\square$

We write the canonical epimorphism $\underline{G} \leftarrow G$ as $\underline{g} \leftarrow\!\shortmid g$. By the lemma, there exists a well-defined function $\alpha : \mathbb{F}^\times \leftarrow \underline{G} \times \underline{G}$ such that, for all $g, h \in G$, we have

$$\widehat{g}\,\widehat{h} = \alpha(\underline{g}, \underline{h})^{-1}\,\widehat{gh}$$

The associativity of $\mathbb{F}Ke$ implies that α is a cocycle. By the lemma again, for each $\underline{g} \in \underline{G}$, there is a well-defined element

$$\widetilde{\underline{g}} = g\,.\,\widehat{g}^{-1}$$

of $\mathbb{F}Ke$. Moreover, $\widetilde{\underline{g}} \in (\mathbb{F}Ge)^K$ because $k\,\widetilde{\underline{g}}\,k^{-1} = kg\,\widehat{g}^{-1}\,\widehat{k}^{-1} = kg\,\widehat{kg}^{-1} = \widetilde{\underline{g}}$. We have

$$\widetilde{\underline{g}}\,\widetilde{\underline{h}} = ghh^{-1}\,\widehat{g}^{-1}\,h\,\widehat{h}^{-1} = gh\,\widehat{h}^{-1}\,\widehat{g}^{-1}\,\widehat{h}\,\widehat{h}^{-1} = gh(\widehat{g}\,\widehat{h})^{-1} = gh(\alpha(g,h)^{-1}\,\widehat{gh})^{-1} = \alpha(g,h)\widetilde{\underline{gh}}\,.$$

Theorem 4.3: *Still assuming that e is G-stable, there is an algebra isomorphism*

$$\tau\ :\ \mathbb{F}_\alpha\underline{G} \to (\mathbb{F}Ge)^K$$

given by $\tau({}_\alpha\underline{g}) = \widetilde{\underline{g}}$ for $\underline{g} \in \underline{G}$. There is also an algebra isomorphism

$$\tau_\otimes\ :\ \mathbb{F}_\alpha\underline{G} \otimes_\mathbb{F} \mathbb{F}Ke \to \mathbb{F}Ge$$

given by $\tau_\otimes(\eta \otimes \xi) = \tau(\eta)\,\xi$ for $\eta \in \mathbb{F}_\alpha\underline{G}$ and $\xi \in \mathbb{F}Ke$. Letting n be the dimension of the simple $\mathbb{F}Ke$-module, we mean to say, $\mathbb{F}Ke = \mathrm{Mat}_n(\mathbb{F})$, then

$$\mathbb{F}Ge \cong \mathrm{Mat}_n(\mathbb{F}_\alpha\underline{G})\,.$$

In particular, the $\mathbb{F}$-algebras $\mathbb{F}Ge$ and $\mathbb{F}_\alpha\underline{G}$ are Morita equivalent.

Proof: By the latest calculation, τ is an algebra homomorphism. Each $\widetilde{\underline{g}} \in (g\mathbb{F}Ke)^K \subseteq g\mathbb{F}Ke$, so the elements $\widetilde{\underline{g}}$ ar linearly independent and τ is injective. We have

$$(g\mathbb{F}Ge)^K = \mathbb{F}\,\widetilde{\underline{g}}$$

because $\widetilde{\underline{g}}^{-1}\zeta \in (\mathbb{F}Ge)^K \cap \mathbb{F}Ke = Z(\mathbb{F}Ke) = \mathbb{F}e$ for all $\zeta \in (g\mathbb{F}Ke)^K$. A comparison of dimensions now confirms that τ is an isomorphism. Noting that $(\mathbb{F}Ge)^K$ and $\mathbb{F}Ke$ commute, it is easy to see that $\tau_\otimes$ is an algebra monomorphism. Again, a comparison of dimensions confirms that $\tau_\otimes$ is an isomorphism. The rider follows immediately. $\square$

Substituting $\overline{N}_G(P, e_P)$ and $\overline{C}_G(P)$ and $\overline{e_P}$ for G and K and e, the latest theorem justifies the final step we made when we were reformulating Alperin's Conjecture in the previous section.

Again, to illustrate the use of the theorem, we shall give an example which is not of the form that arose in our discussion of Alperin's Conjecture. Instead, we shall explain an empirical observation that we made concerning the case where $p = 3$ and $G = \mathrm{GL}_2(3)$. We put $K = Q_8$ and, in the notation that we used before, we put $e = b = z - 1$. Thus, e is the block of $\mathbb{F}K$ associated with the unique 2-dimensional simple $\mathbb{F}K$-module. Above, we observed that the indecomposable projective modules of $\mathbb{F}Ge$ look similar to the indecomposable projective modules of $\mathbb{F}S_3$. Let us now supply a rationale for that observation.

Since $\underline{G} \cong S_3$, the latest theorem says that

$$\mathbb{F}Ge \cong \mathrm{Mat}_2(\mathbb{F}_\alpha S_3)$$

for some twist $[\alpha] \in H^2(S_3, \mathbb{F}^\times)$. The 2-dimensional simple $\mathbb{F}K$-module is realizable over the finite field $\mathbb{F}_3$ and it extends to the natural representation of $\mathbb{F}_3 G$. So there is an isomorphism $\mathbb{F}Ke \cong \mathrm{Mat}_2(\mathbb{F})$ whereby each element $\widehat{g}$ of $\mathbb{F}Ge$ correspondes to g as a 2×2 matrix. Therefore each $\alpha(\underline{g}, \underline{h}) = 1$ and $[\alpha]$ is trivial. We conclude that $\mathbb{F}Ge$ is Morita equivalent to $\mathbb{F}S_3$.

Appendix: An intuitively motivating fantasy

Just as a possible aid to intuition, I offer a fantasy interpretation. Using the finiteness of the twist group $H^2(G, \mathbb{F}^\times)$, together with the algebraic closure of $\mathbb{F}^\times$, it can be shown that the automorphisms $\widehat{g}$ can always be chosen such that all the values of α are roots of unity. Physicists and engineers, of course, usually understand *phase* to refer to an angular shift $e^{i\theta}$. But the group consisting of those complex numbers is the closure of the group of roots of unity. So it makes sense to see α as a system of phase factors.

Under the Copenhagen Interpretation of quantum mechanics, all observable properties of a state remain unchanged when the state vector is multiplied by a phase. Two of the main contributors to that outlook, Heisenberg and Bohr, were influenced, in the 1920s, by the positivism of the Vienna Circle. For hard-line positivists such as Schlick, knowledge of reality was deemed to have no scope beyond that which can be observed; everything else was to be dismissed as "metaphysics".

In particular, phase factors were to be regarded just as mathematical phantoms, with no physical significance. That point of view fitted well enough with Maxwell's use of the quaternions, in electrodynamics, as a calculus of 3-dimensional rotations. There, the $\pm$ signs could readily be regarded as phantoms that have no physical meaning. Bur the positivist position was trickier to maintain when challenged to interpret Dirac's electrodynamics which, arguably, made a more physically substantial use of those $\pm$ signs.

We can now coherently introduce some fantasy historical colour. If we imagine that Clifford theory is a mathematical model for some kind of physical domain, then we can also imagine that, once upon a time, some positivists were insisting that Clifford theory was to be seen as just a phenomenological model for something physically substantial with attributes including G and $\underline{G}$. For them, the twist α was to be seen as an insubstantial ghost, mathematically convenient but with no physical significance. The point of the fantasy is to pretend that α is something non-classical whose ontological status has been controversial.

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