Why are Aristotle and Euclid so Modernistic?

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Abstract: We shall comment on two great transformations in human thought, both of them stimulated partly by interactions between pure mathematics and pure philosophy. Those interactions were, in turn, partly stimulated by metaphysical troubles in mathematics. We shall concentrate mostly on how Aristotle and Euclidian stream responded to those troubles — and to each other — especially as regards the art of definition. Where data is lacking, we shall have to make some comparisons with the philosopher-mathematicians of the late 19th century.

Slightly more than half this talk will concern a dialogue between Aristotle and a stream of thinkers which culminated in Euclid. I shall also be touching upon another story, but somewhat similar, which took place more recently. Let me begin by indicating a motive for narrating such stories in the first place.

When scientists get into trouble, they sometimes reassess their methods, and they look back towards the fundamental. This activity is philosophy. The future being unknown, and the present being a wisp that vanishes before one can comment on it, the data for the philosophy of science resides in the past: what A pointed out five minutes ago, what B pointed out in a seminar this morning, what C pointed out last year, what D pointed out ten years ago, what E pointed out a hundred years ago. But this data can easily be misunderstood, because none of them, B, C, D, E, were aware of what A said five minutes ago. So the data from the past has to be processed before it can be interpreted. The processing of this data is another activity, called history.

This composite discipline, history-and-philosophy, applies very well to the natural sciences, especially physics. At present, the theoretical physicists are in trouble, and some of them are worried that, over the last three or four decades, their community may have produced little work of lasting value. But at least their discussions of the matter are educated by systematic studies of the fundamental principles of their methods. The names of some professional philosophers — Karl Popper, Thomas Kuhn, Paul Feyerbrand — do occasionally crop up in their debates.

Of all the sciences, mathematics has usually been the one most closely tied to pure philosophy, or *protes philosophia* in the sense of Aristotle. However, during the middle of the 20th century, the mathematicians evolved into creatures which, by and large, became incapable of systematic self-reflection. Speaking as a mathematician myself, I guess that, if we were in trouble, we probably would not know it. We have no theory that would serve as a background for examining methodology or for assessing quality. The philosophers of mathematics — such as Imre Lakatos, Hilary Putman, Thomas Tymoczko — rarely crop up in our conversations.

Understanding mathematics to be, so to speak, the sum of its past, I shall be commenting on two great transformations which lie at the centre of what mathematics is. The second of these transformations stretches from the middle 19th to the early 20th century. The first stretches neatly from the early to the late 4th century BC. These two transformations have some common features. Both of them pertained not just to mathematics but to the whole of human thought: to all of science and technology, to all of art and entertainment, and even to religion and politics. Both of them were responses to threats which seemed to undermine a sense of cosmic order; the sense of a rationally arranged or divinely organized universe. In the mathematical component of these transformations, the stimulus arose partly from some counter-intuitive troubles with some of the most basic concepts in mathematics.

The two transformations, of course, were different. During the second transformation — the recent one — a sense of cosmic order was, to some extent, relinquished. Previously, the universe had seemed to be good and right and well-organized, except for the perversity of human beings. But now, it was the universe that was crazy without limit, while human beings were constrained by sanity. In the visual arts, architecture and design, the shift in outlook was called modernism, and part of its emphasis was on breaking away from preconceptions, and on making progress through that which was simple, creative and sometimes counter-intuitive. This same shift of outlook pertained also to all the sciences, including mathematics. The choices of terminology indicate an attraction to the perception of the universe as crazy and chaotic: Heisenberg's Uncertainty Principle, Gödel's Incompleteness Theorems.

In contrast, during the first transformation — the ancient one — a sense of cosmic order was challenged but the response was to reinforce it. The technical term *logos* as it was used in mathematics, indicates a desire to express a victory of reason over mystery.

We shall get onto the mathematical *logoi* in a moment, but first let me touch upon the mathematics of the 19th century. A naive approach to understanding modern mathematics might begin with the question: how do mathematicians view the natural numbers, 0, 1, 2 and so on? One might begin by examining the modern definitions, say, through their set-theoretic constructions or through the Peano Axioms. And then the first question must surely be: why do mathematicians bother with such gnomic definitions of such intuitively plain concepts? The answer, briefly, is that easy definitions are just side-products of difficult definitions.

Let us indicate one sub-plot in the story of the second transformation of mathematics. Some counter-intuitive problems concerning the theory of vibrating strings were brought to a head in 1808 when Fourier developed the theory and applied it to thermodynamics. The methods of Fourier analysis, though, could also be used to obtain conclusions that were plainly false. Much else in analysis, too, was based on shaky geometric perceptions. These troubles were partially resolved by Riemann, who initiated a new conception of geometry and also a new unification of geometry and analysis. It is worth noting that Riemann drew much from his background in theology and philology. His core ideas were set out in his 1841 Habilitation thesis wherein two of the three sections were essentially philosophical or metaphysical. His thesis acknowledges a debt to only two earlier thinkers. One of them, Gauss, was the chief examiner for the thesis. The other, Herbart, was a philosopher working in — or rather, against — the Kantian conception of space.

But Riemann's unification of geometry and analysis was grounded on the notion of the continuum — the notion of the real numbers — yet the continuum was still perceived geometrically, as the number line. The whole castle was still floating on air. Cantor came face-to-face with this problem while developing Riemann's approach to Fourier analysis. He tackled the problem by proposing a definition of the continuum in 1868. Two other definitions of the continuum were put forward by Weierstrauss and by Dedekind in 1871. The continuum was now defined in terms of arithmetic, the natural numbers.

Cantor needed more than this. His applications in analysis required two kinds of infinite number, called ordinal numbers and cardinal numbers. Nowadays, cardinal numbers are discussed in a clinical way at the beginning of undergraduate courses in pure mathematics, but Cantor's papers have a very different flavour. He was an expert theologian, and he was drawing heavily from Spinoza and from the philosophical work of Leibniz.

And what are the natural numbers? Well, they are little more than a pedagogical footnote.

In 1889, Dedekind introduced a cunning definition of the term *finite*, and he realized the natural numbers as the finite ordinal numbers and also as the finite cardinal numbers. He showed that the natural numbers satisfy five conditions, and that all of arithmetic can be derived from those five conditions. A year later, Peano took those five conditions as the definition of the natural numbers, and they are, for that dubious reason, called the Peano Axioms.

Let me reiterate the conclusion. The boring definitions are just side-products of the interesting definitions. The machinery is eventually applied to trivial material, almost as a pedagogical exercise, but it is the material with heavy content that first motivates the creation of the machinery in the first place.

We can now turn to Euclid. It would certainly be a mistake to begin at *Book 1* of *Elements*, because the manuscript sources differ wildly here. Naively, a good place to begin might be *Book 7*, where Euclid defines the natural numbers. Or rather, since zero was not included, I should say the positive integers 1, 2, 3 and so on. *Book 7* opens with "Monad is that by virtue of which each of the things that exist is called one. Arithmos is a multitude composed of units." Something similar can be found in Aristotle's *Metaphysics, Book 14*, "Unit... means that which is quantitatively indivisible". In *Metaphysics, Book 7*, we are informed that, according to some people, "*Arithmos* is a composition of units". Returning to *Elements, Book 7*, and reading on, we find that all of the subsequent definitions, propositions and proofs make sense if we interpret the *monad* to be the number 1, and the *arithmoi* to be the other positive integers.

However, that would be a slight misinterpretation. And besides, such a banal reading of *Elements, Book 7* goes nowhere towards answering the question: why did Aristotle and Euclid have such a neurotic fixation about defining such trivial concepts?

I would suggest that the right place to start reading the *Elements* is the place where Euclid is most in need of his definitions. This is in *Book 5*, the centre-piece of *Elements.* It concerns the most creative piece of mathematics that the classical Greeks produced: the theory of ratios. Their word for ratio is *logos.* He defines

> "*Logos* is a sort of relation in respect to size between two magnitudes of the same kind."

Then he gives a criterion for the existence of a ratio.

> "Magnitudes are said to have *logos* to one another which are capable, when multiplied, of exceeding one another."

In other words, two magnitudes A and B are are said to have a ratio, called the ratio of A to B, if some multiple of A exceeds B and some multiple of B exceeds A. He then gives a criterion for two given ratios to be equal to each other.

> "Magnitudes are said to be in the same *logos*, the first to the second and the third to the fourth, when, if any equimultiples be taken of the first and third, and any equimultiples be taken of the second and fourth, then the former equimultiples are both greater than, or both equal to, or both less than the latter equimultiples."

This is modern mathematics. In effect, he is saying that two ratios of arbitrary magnitudes are equal to each other if and only if they have the same behaviour when compared with ratios of positive integers. This is essentially equivalent to Dedekind's way of constructing the real numbers from the rational numbers. Euclid then gives a criterion for one ratio to be greater than another ratio, and this is essentially Dedekind's definition of the ordering of the real numbers. Dedekind was an expert commentator on *Elements*, and we may surmise that, in fact, Dedekind got the idea from *Elements*. It is tempting to understand the ratios to be the positive real numbers.

To the modern eye, though, the theory is incomplete, because neither Euclid nor any of his successors considers an addition operation on the ratios. But what is interesting is that, even to a discerning ancient eye, the theory is still incomplete. In *Book 8*, Euclid makes use of an operation called $\delta\iota\pi\lambda\alpha\sigma\iota\omega\nu$, or duplication, which corresponds to multiplication of positive real numbers. But, intriguingly — most uncharacteristically — he neglects to define this operation. Virtually all of the classical Greek mathematicians make much use of this operation, but none of them makes any serious attempt to define it. The task of formulating such a definition, in classical Greek terms, would not be straightforward, but it would certainly not be beyond the powers of a Euclid.

We shall return to this mystery in a moment, but first let us say something about the motive for the theory of ratios. In the 5th century BC, and perhaps much earlier, it was already known that the diagonal of a square and the side of a square do not form a *logos* — a ratio — of two integers. The diagonal and the side were said to be incommensurable, they have no common measure: they cannot both be expressed as multiples of some common unit of length. In modern terms, this phenomenon is called the irrationality of the square root of two, but we should not immediately leap to the surmise that the classical Greeks perceived the result in a numerical way. The term for such a situation was *alogos*, in English, irrational. The 5th century mathematician Democritas wrote a book, now lost, on irrational curves and solids. In a letter from Archimedes to Eratosthenes, we hear that Democritus *discovered* the theorem asserting that a cone has one third of the volume of a cylinder with the same height and base. Archimedes opinions, however, that Eudoxus was the first to *prove* this theorem. Eudoxus was a contemporary of Plato. The valid proofs of that theorem are based on a technique called the method of exhaustion, and various much later writers, such as Pappus, tell us us that this method was developed by Eudoxus. I omit discussion of some further clues in Plato and Aristotle which enable us to trace the gradual development of the theory of ratios.

The indications, though, seem to support the following picture: some theorems on volumes must have been discovered (or, at least, discussed) by Democritus, who proved these results using discrete methods that provided arbitrarily accurate approximations. Plato, perhaps though the influence of mathematicians such as Archytas, tried to adapt the mathematical style of reasoning to other areas of philosophy. This approach to philosophy, of course, became established, in his school, the Academy. At this point, the discrete methods of Democritus may have collided with the incommensurability of the diagonal, and also with two paradoxes of Zeno. The more rigorous approach, the method of exhaustion, was pioneered by Eudoxus, who briefly attended Plato's Academy. The development of the abstract and general theory then evolved gradually, apparently over the course of a few decades.

One of the outcomes was that the term *alogos* could now be abandoned. Any two magnitudes of the same kind could now be compared by means of their ratio, their *logos*. In particular, the diagonal D and the side S of a square have a ratio, D to S. It may be that *logos*, here, is to be understood simply as *word* or *term*: the *logos* D to S is a term that can validly be used in mathematical debate. But the numinous connotations were doubtless deliberate: the principle of rational discourse, the divine principle of the cosmos.

The theory of ratios is quite subtle, and it deals scrupulously with questions of existence. To put the matter in modern terms: upon seeing a proof of the irrationality of root two, a natural first reaction is to conclude that the square root of two does not exist. Likewise, some shadow of doubt seems to hang over the existence of these new *logoi*, such as the *logos* of the diagonal to the side. Having mastered the art of arguing these matters in such a careful way, it seems that the mathematicians turned to apply that art to other kinds of mathematical objects. The first proposition in *Elements, Book 1* can be interpreted as a proof that, given two points, then their mid-point exists; it exists, because it can be constructed.

The definition of the positive integers can now be seen in a different light. In both *Physics* and *Metaphysics*, Aristotle stresses that numbers exist, not in the pure sense of Plato's heaven, but in the sense that numbers are numbers of something. In *Physics, Book 4*, he seems to be suggesting that the *arithmos* of a hundred horses is the same as the *arithmos* of a hundred people, but the unit, the *monad*, is different, in one case of horses, in the other case of people. Can there really be two different *monads*, two different numbers 1? Euclid seems to be cunningly avoiding the question. Speculatively, I wonder if Euclid's cunning plan might have been this: the Platonist and the Aristotelian can both be satisfied as they read Euclid's definitions and his subsequent propositions and proofs. They may interpret everything differently, yet they will still both agree that Euclid has everything right.

One might imagine that, enthralled by the spectacular success of the mathematicians, Athenians now flocked to listen to the philosophers, for instance, at Aristotle's *Lyceum*, established in about 335 BC. If they wished to avoid straining too much over tough details, then they might have been keen to listen to something similar in some gentler area, such as ethics or theology. Well, I may be exaggerating here. But still, it does seem that Aristotle — as well as Plato — modelled his general approach on mathematics.

In fact, it might be said that Aristotle, and Plato too, were quite extreme in their emulation of mathematics. Let us recall that Aristotle distinguishes between two kinds of knowledge: *techne*, for the professional classes, and *episteme*, suitable for the aristocrats. Plato, in *Republic, Book 7*, makes a similar distinction between practical arithmetic and pure arithmetic, likewise between practical geometry and pure geometry. Both kinds of knowledge make use of observation and reason. But *episteme*, as it is described and as it is used in Aristotle's *Physics* and *Metaphysics*, seems to have the very peculiar feature that, once the observations have been made, the conclusions then follow by deduction. Normally, not only in science but also in mundane day to day learning about the world, people test their conclusions by making further observations. Why should this process of testing or experiment receive so little emphasis in *episteme*? It is no answer to say that *episteme* was for aristocrats: how did the notion first arise that pure or aristocratic knowledge should be deductive? I am stating the obvious if I merely point out that Aristotle and Plato must have been influenced by the kind of mathematics that we find in Euclid. Speculatively, one could make the stronger suggestion that perhaps the whole character of their philosophy was largely shaped by that kind of mathematics.

At the end of *Metaphysics, Book 1*, Aristotle indicates three examples of the kind of query that might lead to a search for wisdom. The first example is his *self-moving puppets*, something of an enigma, and they appear again in his work *On the Movement of Animals.* The second example is about the timing of the solstices, and this might reasonably be proposed as the beginning of astronomy, since it is fundamental to calibrating the annual cycle. But he gives the most attention to his third example, the irrationality of the diagonal, indeed, he expends some space talking about the wonder of this result, and about how wisdom eventually reverses the sense of wonder over such surprising discoveries. He ends this line of musing with the comment "... for as geometer would wonder at nothing so much than if the diagonal were to become measurable." Perhaps we should take him seriously here. Perhaps he really does mean to suggest that the irrationality of the diagonal lies at the beginning of non-trivial philosophy.

Finally, let us get back to the loose-end above, which is also the loose-end we mentioned in *Elements*. From Euclid onwards, the classical Greeks had three separate concepts which we would nowadays view numerically. Firstly, there were the positive integers. Secondly, there were the magnitudes, which could be added together but not multiplied. Thirdly, there were the ratios, which could be multiplied together but not added. When Archimedes and all the others wished to do both addition and multiplication, they had to constantly convert back and forth between magnitudes and ratios.

Some historians, such as David Fowler and Ian Mueller, have suggested that this clumsy state of affairs arose because the classical Greeks had no numerical perception of ratios. I believe that I have much evidence to counter this, but I will not go into it here. But, at root, my feeling is that Euclid and his predecessors deliberately suppressed the numerical interpretation of ratios precisely because they wished to be rigorous, and they were worried about questions of existence. In a similar way, infinitesimals were suppressed from analysis during the 19th century.

I would go further, and I would suggest that Euclid was the last of the fundamentally creative mathematicians among the classical Greeks. Their failure to develop the material in *Elements, Book 5* was not because of the historian's over-used principle of anachronism; it was just that the classical Greeks stopped doing fundamental research. This may seem to be a contentious proposal, since Archimedes, of the 3rd century, is generally regarded as one of the greatest mathematicians who ever lived, sometimes held to be the greatest of all. But I have some support here from Bourbaki (the famous collective of leading French mathematicians in the middle of the 20th century). Bourbaki marvels at the way Archimedes consistently fails to generalize. Rather than establish a principle, Archimedes frequently repeats variants of the same argument over and over again. Bourbaki describes Archimedes as "the very opposite of systematic".

The collapse of creative mathematics, at the end of the 4th century, may have been caused, perhaps, by a transition away from oral communication towards written communication. When mathematicians speak to each other, we explain the ideas. When we write, we encrypt the ideas as definitions, propositions and proofs. It has sometimes been suggested that Archimedes and Apollonius were so good they killed mathematics, not to be resurrected for a thousand years. No-one else could touch them. I cannot help but wonder whether it might have been Euclid who killed mathematics: he laid down the principles on papyrus, but as if in stone. The principles were no longer to be questioned, nor to be extended.

Whatever the cause of the collapse may have been, the fact is that philosophy and mathematics divorced, and went their separate ways. For classical Greek mathematics, it was the end of history, in the sense of Fukuyama. Except that mathematics and history are activities, and Fukuyama should have called his book "The End of Politics and the Last Politician". So I should be saying: Euclid was the end of classical Greek mathematics.

And that brings us back to the present. We, too, find ourselves in the wake of a great transformation which took place a hundred years ago. Following a surfeit of philosophy, we have abandoned philosophy. So let me end with the unavoidable question: Has mathematics come to an end, again?