MATH 323: Algebra I. **Midterm 2.** LJB, 9 December 2016, Bilkent University.

Time allowed: 105 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question sheet home.

1: 14 marks. Give an example of a group G and subgroups $H \leq G \geq K$ such that $|H| = |K|$ and H is not isomorphic to K. (Make sure it is clear why, in your example, H and K are not isomorphic to each other.)

2: 30 marks. Let G be the group with order 21 such that G has generators a and b satisfying $a^7 = b^3 = 1$ and $bab^{-1} = a^2$. Find the conjugacy classes of G. For each conjugacy class $[g]$, evaluate $|C_G(g)|$.

3: 32 marks. Let G be a finite group, let X be a transitive G-set, and let $x \in X$.
(a) Consider the stabilizer $G_x = \{g \in G : gx = x\}$. Show that G_x is a subgroup of G.
(b) State the Orbit-Stabilizer Equation relating $|G|$ and $|G_x|$.
(c) Show that, given $g, h \in G$, then $gx = hx$ if and only if we have an equality of left cosets $gG_x = hG_x$.
(d) Using part (c), prove the Orbit-Stabilizer Equation.

4: 24 marks. A **24-cell** is a 4-dimensional convex polytope, in other words, it is like a Platonic solid but it is realized in 4-dimensional Euclidian space $\mathbb{R}^4$. A point x in $\mathbb{R}^4$ has the form $x = (x_1, x_2, x_3, x_4)$ where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. The distance between two points $x, y \in \mathbb{R}^4$ is

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + (x_4 - y_4)^2}\ .$$

A **rotation** in $\mathbb{R}^4$ is a distance-preserving function $g : \mathbb{R}^4 \to \mathbb{R}^4$ such that, in a physically intuitive sense, g can be effected by continuous movement. (More precisely but harder to understand: for each $t \in [0, 1]$, there is a distance-preserving function g_t such that, for each $x \in \mathbb{R}^4$, the function $t \mapsto g_t(x)$ is continuous and $g_0(x) = x$ and $g_1(x) = g(x)$). The 24-cell has 24 vertices. The vertices can be expressed as coordinate vectors in two ways:

• we can take the vertices to be the 8 points $(\pm1, 0, 0, 0)$, $(0, \pm1, 0, 0)$, $(0, 0, \pm1, 0)$, $(0, 0, 0, \pm1)$ together with the 16 points $(\pm1/2, \pm1/2, \pm1/2, \pm1/2)$,
• alternatively, we can take the vertices to be the points z such that exactly two of z_1, z_2, z_3, z_4 belong to $\{\pm1\}$ and the other two are 0. For instance, three of the 24 points are $(1, 1, 0, 0)$ and $(1, 0, -1, 0)$ and $(0, -1, 0, -1)$.

Let G be the group of rotational symmetries of the 24-cell. Let x be a vertex and ϵ an edge of the 24-cell.
(a) Evaluate $|G_x|$, hence evaluate $|G|$.
(b) Two vertices x and y have an edge between them if and only if x and y are distinct and the distance $d(x, y)$ is as small as possible. Without using part (a), determine the number of edges, evaluate $|G_\epsilon|$, hence check your evaluation of $|G|$.
(c) What is the isomorphism class of G_x?
(d) Let g be an element of G_x with order 3. What is the size of the conjugacy class $[g]_G$ of g in G?
(e) Show that there is a unique element $f \in G$ such that f has order 2 and $fg = gf$.
(f) What is the isomorphism class of G_ϵ? (Hint: part (e) might be useful.)

Midterm 1 Solutions

There is no such thing as a "model solution". Often, there are many good ways of deducing a given conclusion.

1: The smallest example is $(G, H, K) = (D_8, C_4, V_4)$. Of course, $C_4 \not\cong V_4$ by considering orders of group elements.

2: We shall show that the conjugacy classes of G are

$$\{1\} \ , \qquad \{a, a^2, a^4\} \ , \qquad \{a^3, a^5, a^6\} \ , \qquad \{a^i b : i \in \mathbb{Z}\} \ , \qquad \{a^i b^2 : i \in \mathbb{Z}\} \ .$$

Conjugating a by b and b^2, we see that $\{a, a^2, a^4\} \subseteq [a]$. On the other hand, $\langle a \rangle \leq C_G(a)$. But $|[a]||C_G(a)| = |G| = 21$. So $|[a]| = 3$ and $|C_G(a)| = 7$. We have shown that $[a] = \{a, a^2, a^4\}$. Similarly, $[a^{-1}] = \{a^{-1}, a^{-2}, a^{-4}\} = \{a^3, a^5, a^6\}$. Since $\langle b \rangle \leq C_G(b) < G$, Lagrange's Theorem gives $|C_G(b)| = 3$, hence $|[b]| = 7$. By considering the quotient group $G/\langle a \rangle \cong C_3$, we see that all the conjugates of b belong to the coset $\langle a \rangle b$. Therefore $[b] = \langle a \rangle b$ and similarly for b^2.

Comment: This question can also be done just as quickly by just calculating the conjugacy classes of a and b directly, without using any theorems, then appealing to similarity for the other elements of G. The above solution illustrates a technique involving the Orbit-Stabilizer Equation. The technique becomes more useful for larger given groups, at least as a check.

3: Part (a). Let $g, h \in G_x$. Then $ghx = gx = x$ and $g^{-1}x = g^{-1}gx = x$, hence $gh \in G \ni g^{-1}$.

Part (b). We have $|G| = |X|.|G_x|$.

Part (c). The condition $gx = hx$ can be expressed as $h^{-1}gx = x$, in other words, $h^{-1}g \in G_x$, which is equivalent to $gG_x = hG_x$.

Part (d). Given $g \in G$ and $y \in X$ then, by part (c), $gx = y$ if and only if every element of gG_x sends x to y. Since X is transitive, that condition describes a bijective correspondence $gG_x \leftrightarrow y$ between the left cosets $gG_x \subseteq G$ and the elements $y \in X$. Therefore, $|X|$ is the number of left cosets gG_x. The required equality follows, because all of those left cosets have size $|G_x|$.

Comment: If we, or the reader, were not aware that all the cosets gG_x have the same size, then we could argue as in the proof of Lagrange's Theorem, directly showing that, for each y, the number of group elements sending x to y coincides with the number of group elements sending x to x.

The above argument also shows that, putting $H = G_x$, then X is a copy of the G-set of left cosets $G/H = \{aH : a \in G\}$, with each group element $g \in G$ sending aH to gaH. Moreover, we can construct the G-set G/H for any subgroup $H \leq G$. Replacing X with G/H, the argument in part (d) and the previous paragraph becomes exactly the proof we gave for Lagrange's Theorem. Thus, in essence, the Orbit-Stabilizer Equation and Lagrange's Theorem are two different ways of expressing the same underlying content.

4: Parts (a) and (c). The 8 neighbouring vertices $(1, \pm1, \pm1, \pm1)$ of $(1, 0, 0, 0)$ comprise a cube whose group of rotations can be identified with G_x. So $G_x \cong S_4$ and $|G_x| = 24$. Since there are 24 vertices, $|G| = 24|G_x| = 576$.

Part (b). Since each vertex has 8 edges, while each edge has 2 vertices, the number of edges is $4.24 = 96$. Without loss of generality, the edge ϵ has vertices $x = (1, 1, 0, 0)$ and

$y = (1,0,1,0)$. Regarding G_x as the rotation group of the cube C of nearest neighbours of x, also noting that y is a vertex of C, observe that the stabilizer of y in G_x is $G_x \cap G_y$. Dividing by the number of vertices of C, we have $|G_x \cap G_y| = |G_x|/8 = 3$. The vertices x and y are interchanged by the rotation that fixes $(0,1,1,0)$ and rotates C through half a revolution about the mid-point of ϵ. Therefore $|G_\epsilon| = 2|G_x \cap G_y| = 6$. We recover the equality $|G| = 96|G_\epsilon| = 576$.

Part (d). The group S_4 has exactly 8 elements of order 3, and they are mutually conjugate. The elements of G with order 3 fixing x and y also fix $(-1,-1,0,0)$, $(-1,0,-1,0)$, $(0,1,-1,0)$, $(0,-1,1,0)$ but cannot fix any other vertices because they cannot fix a 3-dimensional subspace. Therefore, each vertex of G is fixed by 8 conjugates of g, while each conjugate of g fixes 6 vertices. So the numbers 24.8 and $6|[g]_G|$ are both equal to the number of pairs (g', z) such that g' is a conjugate of g fixing vertex z. Therefore $|[g]_G| = 32$.

Part (e). A group element of order 2 is called an **involution**. Let f be the involution in G sending each vertex (t,u,v,w) to its opposite vertex $(-t,-u,-v,-w)$. Then $f \in Z(G)$ and, in particular, $fg = gf$. By part (d), $|C_G(g)| = |G|/32 = 18$. The group Z generated by f is a normal subgroup of $C_G(g)$, and $|C_G(g)/Z| = 9$, which is odd. By Lagrange's Theorem, $C_G(g)/\langle f \rangle$ has no involution. Therefore f is the unique involution on $C_G(g)$.

Part (f). By part (b), $G_\epsilon \cong C_6$ or $G_\epsilon \cong S_3$. We may assume that $g \in G_\epsilon$. Plainly, $f \notin G_\epsilon$. Hence, via part (e), no involution in G_ϵ commutes with g. Therefore, $G_\epsilon \cong S_3$.

Comment: In the question, the equivalence of the two coordinatizations of the vertices was merely stated. The two coordinatizations do describe the same polytope, indeed, there is a linear map from the first to the second such that

$$(1,0,0,0) \mapsto (1,1,0,0)/2 \, , \qquad (0,1,0,0) \mapsto (1,-1,0,0)/2 \, ,$$

$$(0,0,1,0) \mapsto (0,0,1,1)/2 \, , \qquad (0,0,0,1) \mapsto (0,0,1,-1)/2 \, .$$

Noting that vertices $(1,0,0,0)$, $(0,1,0,0)$, $(1,1,\pm 1,\pm 1)/2$ form an octahedron with centroid $(1,1,0,0)/2$, we see that the vertices of either one of two specified polytopes are, as vectors, double the centroids of the octahedral faces of the other specified polytome. So the 24-cell is self-dual.