# MATH 227: Introduction to Linear Algebra. Fall 2018. Midterm 2

LJB, 3 December 2018, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

**1: 25 marks.** For each of the following real vector spaces $V$, find a basis for $V$ and evaluate $\dim(V)$:

**(a)** $V = \mathbb{R}^5$.

**(b)** $V = \text{span}\{(1,1,1,1,1),(1,3,5,7,11),(3,5,7,9,13),(3,5,7,11,13),(7,13,19,25,37)\}$ as a subspace of $\mathbb{R}^5$.

**(c)** $V$ is the vector space whose vectors are the linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$.

**2: 25 marks.** For each of the following subsets $U$ of a real vector space $V$, show that $U$ is a subspace of $V$, find a basis for $U$ and evaluate $\dim(U)$.

**(a)** $V = \text{Mat}_2(\mathbb{R})$ (the real vector spaces whose vectors are the $2 \times 2$ matrices with entries in $\mathbb{R}$), and $U$ is the set of matrices $A$ in $V$ such that $A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$.

**(b)** $V$ is as in part (c) of Question 1 and $U$ is the set of linear transformations $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\theta((1,0)) \in \text{span}\{(1,0)\}$.

**3: 25 marks.** Consider a Markov process with three states. The transition probability from State 1 to State 2 is equal to the transition probability from State 1 to State 3. If the system is in State 1 or 3 at time $t$, then the probability of remaining in that state at time $t+1$ is $1/2$. If it is in State 2 at time $t$, then the probability of remaining in that state at time $t+1$ is $1/4$. The system cannot move from State 2 or State 3 to State 1. Suppose the system is in State 1 at time $t = 0$. What is the probability of the system being in State 3 at time $t = 6$? (For full marks, express your answer in a simple form.)

**4: 25 marks.** Let $X$ and $Y$ be subspaces of a finite-dimensional real vector space $V$.

**(a)** We define $X \cap Y$ to be the set of vectors $v$ such that $v \in X$ and $v \in Y$. Show that $X \cap Y$ is a subspace of $V$.

**(b)** We define $X + Y$ to be the set of vectors having the form $x + y$ where $x \in X$ and $y \in Y$. Show that $X + Y$ is a subspace of $V$.

**(c)** Show that there exists a basis $\mathcal{B}$ for $V$ such that $\mathcal{B}$ contains bases for all four subspaces $X$, $Y$, $X \cap Y$, $X + Y$.

**(d)** Find a formula expressing $\dim(X + Y)$ in terms of $\dim(X)$, $\dim(Y)$, $\dim(X \cap Y)$. (Hint: to prove your answer, part (c) may be helpful.)

# Midterm 2 Solutions

**1:** Part (a). The standard basis

$$\{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1)\}$$

is a basis for $V$. In particular, $\dim(V) = 5$.

Part (b). To examine the linear relations between the given vectors,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 & 11 \\ 3 & 5 & 7 & 9 & 13 \\ 3 & 5 & 7 & 11 & 13 \\ 7 & 13 & 19 & 25 & 37 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 & 10 \\ 0 & 2 & 4 & 6 & 10 \\ 0 & 2 & 4 & 8 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To obtain the second matrix, we added multiples of the first row to the other rows. To obtan the third, we added multiples of the second row to the third, fourth, fifth rows, then divided two rows by 2 and interchanged two rows. Evidently, $\{(1,1,1,1,1),(0,1,2,3,5),(0,0,0,1,0)\}$ is a basis for $V$ and and $\dim(V) = 3$.

Part (c). Let $\alpha_{1,1}$, $\alpha_{1,2}$, $\alpha_{2,1}$, $\alpha_{2,2}$, respectively, be the linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ with respect to the standard basis $\{(1,0),(0,1)\}$. Then $\{\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2}\}$ is a basis for $V$. Hence, $\dim(V) = 4$.

**2:** Part (a). Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$, then $A \in U$ if and only if $\begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ b & d \end{bmatrix}$, that is, $c = 0$ and $a = d$. Hence $U = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$. Since $U$ is the span of a set, $U$ is a subspace. Plainly, $U$ has basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ and $\dim(U) = 2$.

Part (b). Given $\theta \in V$, then $\theta \in U$ if and only if $\theta$ is represented by a matrix having the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $a, b, c \in \mathbb{R}$. So, in the notation of part (c) of Question 1, $U = \text{span}\{\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,2}\}$. In particular, $U$ is a subspace of $V$. Furthermore, $U$ has basis $\{\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,2}\}$ and $\dim(U) = 3$.

**3:** The Markov matrix for the process is $M = N/4$ where $N = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$.

We have $N^2 = \begin{bmatrix} 4 & 0 & 0 \\ 2+1+2 & 1+6 & 2+4 \\ 2+3+2 & 3+6 & 6+4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 7 & 6 \\ 7 & 9 & 10 \end{bmatrix}$

and $N^4 = \begin{bmatrix} 16 & * & * \\ 20+35+42 & * & * \\ 28+45+70 & * & * \end{bmatrix} = \begin{bmatrix} 16 & * & * \\ 97 & * & * \\ 143 & * & * \end{bmatrix}$

where $*$ indicates an unspecified entry. The $(3,1)$ entry of $N^6$ is

$$16.7 + 97.9 + 143.10 = 112 + 873 + 1430 = 2415\ .$$

The answer is the $(3,1)$ entry of $M$, which is $2415/4^6 = 2415/4096$.

**4:** Part (a). Let $v_1, v_2 \in X \cap Y$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Write $v = \lambda_1 v_1 + \lambda_2 v_2$. We are to show that $v \in X \cap Y$. We have $v \in X$ because $X$ is a subspace. Similarly, $v \in Y$. Therefore, $v \in X \cap Y$.

Part (b). Let $v_1, v_2 \in X + Y$ and $\lambda_1, \lambda_2$ as before. Write $v_1 = x_1 + y_1$ and $v_2 = x_2 + y_2$ with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then

$$\lambda_1 v_1 + \lambda_2 v_2 = (\lambda_1 x_1 + \lambda_2 x_2) + (\lambda_2 y_1 + \lambda_2 y_2) \in X + Y \ .$$

Part (c). We make use of the fact that, given a subspace $K$ of a finite-dimensional real vector space $L$, then any basis $\{s_1, ..., s_m\}$ of $K$ extends to a basis $\{s_1, ..., s_n\}$ of $L$, with $m \leq n$.

Let $\{e_1, ..., e_a\}$ be a basis for $X \cap Y$. Extend to a basis $\{e_1, ..., e_a, f_1, ..., f_b\}$ of $X$ and to a basis $\{e_1, ..., e_a, g_1, ..., g_c\}$ of $Y$. We claim that the set $\mathcal{A} = \{e_1, ..., e_a, f_1, ..., f_b, g_1, ..., g_c\}$ is a basis for $X + Y$. Plainly, $\mathcal{A}$ spans $X + Y$. To show linear independence, consider a linear combination

$$\sum_i \lambda_i e_i + \sum_j \mu_j f_j + \sum_k \nu_k g_k = 0 \ .$$

Then $\sum_k \nu_k g_k \in X \cap Y$, so $\sum_k \nu_k g_k = \sum_i \lambda'_i e_i$ for some coefficients $\lambda'_i$. Therefore

$$\sum_i (\lambda_i + \lambda'_i) e_i = \sum_j \mu_j f_j \ .$$

By the linear independence of the above basis for $Y$, each $\mu_j = 0$. Similarly, each $\nu_k = 0$. The linear independence of the above basis for $X \cap Y$ now implies that each $\lambda_i = 0$. We have shown that $\mathcal{A}$ is linearly independent, and the claim is established.

Finally, we can extend the above basis for $X + Y$ to a basis

$$\mathcal{B} = \{e_1, ..., e_a, f_1, ..., f_b, g_1, ..., g_c, h_1, ..., h_d\}$$

for $V$. Plainly, $\mathcal{B}$ contains the above bases for $X$, $Y$, $X \cap Y$, $X + Y$.

Part (d). In the notation of part (c),

$$\dim(X + Y) = a + b + c = (a + b) + (a + c) - a = \dim(X) + \dim(Y) - \dim(X \cap Y) \ .$$