MATH 227: Introduction to Linear Algebra. Midterm 1

LJB, 9 March 2017, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 15 marks. Using Gaussian elimination, solve the equations

$$2x + y + 4z = 3x + 5y + 7z = 6x + 9y + 8z = 1 \ .$$

2: 25 marks. Consider the matrix $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$.

(a) Using the Gauss–Jordan method, find A^{-1}.

(b) Check your answer to Question 1 using the matrix A^{-1}.

3: 25 marks. Let A be as in Question 2.

(a) Evaluate $\det(A)$.

(b) Check your answer to the first part of Question 2 by calculating A^{-1} using the cofactor method.

4: 20 marks. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix such that, for all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$, we have

$$(ax + by)^2 + (cx + dy)^2 = x^2 + y^2 \ .$$

(a) Show that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1$, we mean to say, the determinant is 1 or -1.

(b) Simplify the expression $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

5: 15 marks. (Recall, the kind of numbers we have been working with, thus far in the course, are called the real numbers. Of course, they can be viewed as the points on the real number line. In this question, we continue to work with the real numbers, just as usual. The question does not involve any other kind of number, such as the imaginary numbers.)

(a) Let A be a 3×3 matrix (with entries in the real numbers, as usual). Show that there exists some t (a real number) and some nonzero vector x (whose coordinates are real numbers) such that $Ax = tx$.

(b) Does the conclusion still hold when we replace A with a 4×4 matrix?

Midterm 1 Solutions

1: The system is $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 3 & 5 & 7 & 1 \\ 6 & 9 & 8 & 1 \end{array}\right]$. Operation $r_2' = 2r_2$ gives $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 6 & 10 & 14 & 2 \\ 6 & 9 & 8 & 1 \end{array}\right]$. Then $r_2' = r_2 - 3r_1$ and $r_3' = r_3 - 3r_1$ give $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 0 & 7 & 2 & -1 \\ 0 & 6 & -4 & -2 \end{array}\right]$. Then $r_2' = r_2 - r_3$ gives $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 0 & 1 & 6 & 1 \\ 0 & 6 & -4 & -2 \end{array}\right]$. Applying the operation $r_3' = r_3 - 6r_2$ gives $\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & -40 & -8 \end{array}\right]$.

So $z = 1/5$ and $y = 1 - 6z = 1 - 6/5 = -1/5$ and $2x = 1 - y - 4z = 1 + 1/5 - 4/5 = 2/5$, hence $x = 1/5$. In conclusion, $(x, y, z) = (1, -1, 1)/5$.

2: Part (a). Starting from $\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 5 & 7 & 0 & 1 & 0 \\ 6 & 9 & 8 & 0 & 0 & 1 \end{array}\right]$, the row operations in Question 1 yield $\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 7 & 2 & -3 & 2 & 0 \\ 0 & 6 & -4 & -3 & 0 & 1 \end{array}\right]$, then $\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 6 & 0 & 2 & -1 \\ 0 & 6 & -4 & -3 & 0 & 1 \end{array}\right]$ then $\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 6 & 0 & 2 & -1 \\ 0 & 0 & -40 & -3 & -12 & 7 \end{array}\right]$. The operation $r_3' = -r_3/40$ yields $\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 40/40 & 0 & 0 \\ 0 & 1 & 6 & 0 & 80/40 & -40/40 \\ 0 & 0 & 1 & 3/40 & 12/40 & -7/40 \end{array}\right]$. Then $r_1' = r_1 - 4r_3$ and $r_2' = r_2 - 6r_3$ give $\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 28/40 & -48/40 & 28/40 \\ 0 & 1 & 0 & -18/40 & 8/40 & 2/40 \\ 0 & 0 & 1 & 3/40 & 12/40 & -7/40 \end{array}\right]$, whence $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 23/40 & -28/40 & 13/40 \\ 0 & 1 & 0 & -18/40 & 8/40 & 2/40 \\ 0 & 0 & 1 & 3/40 & 12/40 & -7/40 \end{array}\right]$ using $r_1' = (r_1 - r_2)/2$.

In conclusion, $A^{-1} = \dfrac{1}{40}\begin{bmatrix} 23 & -28 & 13 \\ -18 & 8 & 2 \\ 3 & 12 & -7 \end{bmatrix}$.

Part (b), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \dfrac{1}{40}\begin{bmatrix} 23 - 28 + 13 \\ -18 + 8 + 2 \\ 3 + 12 - 7 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

3: Part (a). We have $\det(A) = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 9 & 8 \end{vmatrix} = 2\begin{vmatrix} 5 & 7 \\ 9 & 8 \end{vmatrix} - \begin{vmatrix} 3 & 7 \\ 6 & 8 \end{vmatrix} + 4\begin{vmatrix} 3 & 5 \\ 6 & 9 \end{vmatrix}$

$= 2(40 - 63) - (24 - 42) + 4(27 - 30) = -46 + 18 - 12 = -40.$

Part (b). The cofactor matrix is $\begin{bmatrix} 40 - 63 & 42 - 24 & 27 - 30 \\ 36 - 8 & 16 - 24 & 6 - 18 \\ 7 - 20 & 12 - 14 & 10 - 3 \end{bmatrix} = \begin{bmatrix} -23 & 18 & -3 \\ 28 & -8 & -12 \\ -13 & -2 & 7 \end{bmatrix}$. Taking the transpose and dividing by $\det(A) = -40$ yields the value calculated for A^{-1} in Question 1.

4: Part (a). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\Delta = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$. We are to show that $\Delta^2 = 1$. Write

$\begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$. Thus, $u = ax + by$ and $v = cx + dy$. The given assumption on A is that $u^2 + v^2 = x^2 + y^2$. Letting (x, y) take the values $(1, 0)$, $(0, 1)$, $(1, 1)$, respectively, then (u, v) takes the values (a, c), (b, d), $(a + b, c + d)$. By the given assumption,

$$1 = a^2 + c^2 = b^2 + d^2 \ , \qquad 2 = (a + b)^2 + (c + d)^2 \ .$$

Expanding the right-hand equation, then using the left-hand equations, we obtain

$$ab + cd = 0 \ .$$

Squaring, $0 = (ab + cd)^2 = a^2b^2 + c^2d^2 + 2abcd$. Meanwhile, $\Delta = ad - bc$. Therefore

$$\Delta^2 = a^2d^2 + b^2c^2 - 2abcd = a^2d^2 + b^2c^2 + a^2b^2 + c^2d^2 = (a^2 + c^2)(b^2 + d^2) = 1 \ .$$

Part (b). We have $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} du - bv \\ av - cu \end{bmatrix}$. Letting (u, v) take the values $(1, 0)$, $(0, 1)$, $(1, 1)$, respectively, then $\Delta(x, y)$ takes the values $(d, -c)$, $(-b, a)$, $(d - b, a - c)$. Arguing as before, again using two equations to modify a third, we have

$$1 = a^2 + b^2 = c^2 + d^2 \ , \qquad ac + bd = 0 \ .$$

Therefore $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Comment: The given assumption on A is that action by A preserves distances between points on the plane. The only matrices with that effect are rotations about the point $(0, 0)$ and reflections across a line passing through $(0, 0)$. An alternative solution, more conceptual, is based on that observation.

We can recover the observation algebraically as follows. Since $a^2 + c^2 = 1$, we have $0 \leq a^2 \leq 1$, hence $-1 \leq a \leq 1$. So there exists some θ such that $a = \cos(\theta)$. It follows that $c = \pm \sin(\theta)$. Replacing θ with $-\theta$ if necessary, we can choose θ such that $c = \sin(\theta)$. It is now not hard to show that

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \text{ or } A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & -\cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

In the former of those two cases, A is a rotation, in the latter, a reflection.

5: Part (a). The equation $Ax = tx$ can be rewritten as $(A - tI)x = 0$. Given t, then there exists a nonzero such x if and only if $A - tI$ is non-invertible, in other words, $\det(A - tI) = 0$. Since A is a 3×3 matrix, there exist β, γ, δ, depending only on A, such that

$$\det(A - tI) = -t^3 + \beta t^2 + \gamma t + \delta \ .$$

That expression is plainly positive for some value $t = a < 0$ and negative for some value $t = b > 0$. By continuity, there exists some c in the range $a < c < b$ such that $\det(A - tI) = 0$.

Part (b). No, the conclusion fails when $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$.