MATH 227: Introduction to Linear Algebra. Fall 2018. Midterm 1

LJB, 5 November 2018, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to justify your answers, except in any cases where your answers are obvious.

1: 15 marks. Using Gaussian elimination, solve the simultaneous equations

$$x + y + z = 4 \,, \qquad x + 2y + 4z = 5 \,, \qquad x + 4y + 16z = 6 \,.$$

2: 40 marks. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}$.

(a) Using the Gauss–Jordan method, calculate the inverse A^{-1}.

(b) Using any method, calculate the determinant $\det(A)$.

(c) Using part (b) and the method of cofactors, again calculate A^{-1}.

3: 15 marks. Let $\underline{x} = (1, 4, 7, 5, 9, 7, 13, 10)$ and $\underline{y} = (4, 0, 5, 7, 5, 7, 8, 12)$ be sample values for two paired variables. Calculate the Pearson correlation coefficient of the sample values.

4: 15 marks. Find the distances between:

(a) the point $(1, 2, 3)$ and the plane consisting of the points (x, y, z) such that $2x+y+2z = 19$.

(b) the point $(1, 2, 3)$ and the line consisting of the points (x, y, z) such that $2x + y + 2z = 19$ and $2x + 2y + z = 21$.

5: 15 marks. Let n be a positive integer. An $n \times n$ matrix M is called a **pseudostochastic matrix** provided the sum of the entries in each column is 1.

(a) Let M and N be pseudostochastic $n \times n$ matrices. Show that MN is pseudostochastic.

(b) Let $\underline{x} = (x_1, ..., x_n)$ and $\underline{y} = (y_1, ..., y_n)$ be n-dimensional vectors such that $\underline{y} = M\underline{x}$. Express $y_1 + ... + y_n$ in terms of the n numbers x_1, ..., x_n. (Remember to justify your answer.)

Midterm 1 Solutions

1: The system is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 5 \\ 1 & 4 & 16 & 6 \end{array}\right]$.

Subtracting row 1 from the other two rows, $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 15 & 2 \end{array}\right]$.

Subtracting 3 times row 2 from row 3 gives $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 6 & -1 \end{array}\right]$.

So $z = -1/6$. Hence $y = 1 - 3z = 3/2$ and $x = 4 - y - z = 24/6 - 9/6 + 1/6 = 8/3$. In conclusion, $(x, y, z) = (8/3, 3/2, -1/6)$.

2: Part (a). We code the problem as $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{array}\right]$. The operations in part (a) give $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 3 & 15 & -1 & 0 & 1 \end{array}\right]$, then $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 6 & 2 & -3 & 1 \end{array}\right]$.

Multiplying row 3 by a factor, $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Adding multiples of row 3, $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2/3 & 1/2 & -1/6 \\ 0 & 1 & 0 & -2 & 5/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Subtracting row 2 from row 1, $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/3 & -2 & -1/3 \\ 0 & 1 & 0 & -2 & 5/2 & -1/2 \\ 0 & 0 & 1 & 1/3 & -1/2 & 1/6 \end{array}\right]$.

Therefore, $A^{-1} = \dfrac{1}{6}\begin{bmatrix} 16 & -12 & 2 \\ -12 & 15 & -3 \\ 2 & -3 & 1 \end{bmatrix}$.

Part (b). In view of the upper triangular matrix in part (a), we have $\det(A) = 6$.

Part (c). The matrix of minors is

$$\begin{bmatrix} 2.16 - 4.4 & 1.16 - 1.4 & 1.4 - 1.2 \\ 1.16 - 1.4 & 1.16 - 1.1 & 1.4 - 1.1 \\ 1.4 - 1.2 & 1.4 - 1.1 & 1.2 - 1.1 \end{bmatrix} = \begin{bmatrix} 16 & 12 & 2 \\ 12 & 15 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

The matrix of cofactors is $\begin{bmatrix} 16 & -12 & 2 \\ -12 & 15 & -3 \\ 2 & -3 & 1 \end{bmatrix}$. Taking the transpose and dividing by the determinant calculated in part (b), we recover A^{-1} as in part (a).

3: The means are $\overline{x} = (1 + 4 + 7 + 5 + 9 + 7 + 13 + 10)/8 = 56/8 = 7$ and $\overline{y} = (4 = 0 + 5 + 7 + 5 + 7 + 8 + 12)/8 = 48/8 = 6$. The centred vectors are

$$\widetilde{x} = (-6, -3, 0, -2, 2, 0, 6, 3)\ , \qquad \widetilde{y} = (-2, -6, -1, 1, -1, 1, 2, 6)\ .$$

We have $\widetilde{x}.\widetilde{y} = 12+18+0-2-2+0+12+18 = 2.28 = 56$. Also,

$$\| \widetilde{x} \|^2 = 36+9+4+4+36+9 = 2.49 = 98\,, \quad \| \widetilde{y} \|^2 = 4+36+1+1+1+1+36+4 = 2.42 = 84\,.$$

The correlation coefficient is $\rho = \dfrac{\widetilde{x}.\widetilde{y}}{\| \widetilde{x} \| \,.\, \| \widetilde{y} \|} = \dfrac{56}{\sqrt{2.49}.\sqrt{2.42}} = \dfrac{56}{2.7.\sqrt{42}} = 4/\sqrt{42}$.

4: Part (a). The distance is $\dfrac{|2.1+1.2+2.3-19|}{\sqrt{2^2+1^2+2^2}} = \dfrac{|-9|}{\sqrt{9}} = 3$.

Part (b). Subtracting the equation of one plane from the equation of the other yields $y - z = 2$. Hence $2x = 19 - y - 2z = -3z + 17$. So the point (x, y, z) lies on the line if and only if

$$(x, y, z) = (-3z/2 + 17/2, z + 2, z)\,.$$

The distance D between (x, y, z) and $(1, 2, 3)$ is given by

$$D^2 = \| (x, y, z) - (1, 2, 3) \| = (x-1)^2 + (y-2)^2 + (z-3)^2\,.$$

Completing the square,

$$4D^2 = (-3z+15)^2 + 4z^2 + 4(z-3)^2 = 17z^2 - 114z + 261 = 17(z - 57/17)^2 + 261 - 57^2/17\,.$$

The required distance d is the minimum value of D. We have

$$17.261 - 57^2 = 17 + 17.4.65 - 4.14.57 - 57 = 4(17.65 - 14.57 - 10)\,.$$

So $17d^2 = 3.65 + 14(65-57) - 10 = 195 + 14.8 - 10 = 185 + 112 = 297$. So $d = \sqrt{297/17}$.

Alternative for part (b): Let π_1 and π_2 be the planes given by the equations $2x+y+2z-19 = 0$ and $2x + 2y + z - 21 = 0$, respectively. Let $P = (1, 2, 3)$. Let P_1 and P_2, respectively, be the closest point on π_1 and π_2 to P. In part (a), we saw that the distance between P and P_1 is 3. The distance between P and P_2 is

$$\frac{|2.1+2.2+1.3-21|}{\sqrt{2^2+2^2+1^2}} = \frac{|-12|}{3} = 4\,.$$

The angle θ between P_1 and P_2 at P is given by

$$\cos(\theta) = \frac{(2,1,2).(2,2,1)}{\| (2,1,2) \| \,.\, \| (2,2,1) \|} = \frac{4+2+2}{3.3} = \frac{8}{9}\,.$$

Let Q be the closest point on the line to P. The points P, P_1, P_2, Q all lie on a plane. We can coordinatize the plane by putting $P = (0, 0)$ and $P_1 = (p, q)$ and $P_2 = (4, 0)$ where $p = 3\cos(\theta)$ and $\| (p, q) \| = 3$. Then $q^2 = 3^2 - p^2 = 9 - 64/9 = (81-64)/9 = 17/9$. So $q = \sqrt{17}/3$. Now Q is the intersection of the line whose points (x, y) satisfy $x = 4$ and the line whose points (x, y) satisfy $(p, q).(x, y) = \| (p, q) \|^2$. So $Q = (x, y)$ where $x = 4$ and $8x/3 + \sqrt{17}y/3 = 9$, in other words, $y = (27-32)/\sqrt{17} = -5/\sqrt{17}$. Finally, the distance d between P and Q is given by $d^2 = x^2 + y^2 = 16 + 25/17 = (272+25)/17 = 297/17$. So $d = \sqrt{297/17}$.

5: Part (a). Let $M_{i,j}$ denote the (i, j)-entry of M, and similarly for N and MN. We have $\sum_i M_{i,j} = \sum_j N_{j,k} = 1$ and $(MN)_{i,k} = \sum_j M_{i,j}N_{j,k}$, hence $\sum_i (MN)_{i,k} = \sum_{i,j} M_{i,j}N_{j,k} = 1$.

Part (b). We have $y_i = \sum_j M_{i,j}x_j$, so $\sum_i y_i = \sum_{i,j} M_{i,j}x_j = \sum_j x_j$.