MATH 210: Finite and Discrete Mathematics. Midterm 1

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

Do not forget: Justify your answers. A proof is a very clear deductive explanation. Arguments will be marked according to how clearly and correctly they would communicate with other members of the class.

All graphs are understood to be finite ordinary graphs.

LJB, 11 March 2016, Bilkent University.

1: 10% Show that $1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2 \leq 2 - 1/n$ for all positive integers n.

2: 20% Let r be a real number such that $r + 1/r$ is an integer. Show that, $r^n + 1/r^n$ is an integer for all positive integers n. (Hint: consider the expression $(r^n + 1/r^n)(r + 1/r)$.)

3: 20% Let c be a real number. Let x_0, x_1, ... be an infinite sequence of real numbers such that $x_{n+2} + 2x_{n+1} + cx_n = 0$ and $x_0 = -1$ and $x_1 = 10$.
(a) In this part of the question, suppose that $c = -8$. Give a general formula for x_n.
(b) In this part, suppose there exist real numbers C, D, γ such that $x_n = (C + nD)\gamma^n$ for all n. Find the possible values of c, C, D, γ.

4: 30% The graph of the cube C_3, sometimes called the 3-cube, can be described as the graph whose vertices are the binary strings 000, 001, 010, 011, 100, 101, 110, 111 where two vertices are adjacent if and only if they differ at exactly one digit. For instance 010 and 110 are adjacent because the differ only in the left-hand digit. The vertices 010 and 100 are not adjacent because they differ at both the left-hand digit and the middle digit.
(a) Does C_3 have an Euler circuit?
(b) Is C_3 a planar graph?
(c) Let C_4 be the graph described similarly but with binary strings of length 4. Thus, for instance, the vertices 0110 and 1110 are adjacent because they differ at only one digit, but 0110 and 1100 are not adjacent because they differ at 2 digits. Draw a diagram of G and find an Euler circuit. (To specify the Euler circuit, list the vertices in order. Numbering the edges is not recommended. If you draw a nasty little diagram with the edges numbered with teeny tiny numbers, the examiner will not strain his eyesight trying to read it.)
(d) Is C_4 a planar graph?
(e) Let C_8 be the graph described as above, but now with binary strings of length 8 (so that there are now 256 vertices and each vertex has 8 edges). Does C_8 have an Euler circuit?
(f) Is C_8 a planar graph?

5: 20% Let G be a graph with n vertices and $(n-1)(n-2)/2 + 1$ edges. Show that G is connected.

Solutions to Midterm 1

Solution 1: Let $A_n = 1/1^2 + ... + 1/n^2$ and $B_n = 2 - 1/n$. We are to show that $A_n \le B_n$ for all positive integers n. The inequality holds when $n = 1$. because $A_1 = 1 = B_n$. Now suppose that $n \ge 2$ and that $A_{n-1} \le B_{n-1}$. We have

$$B_n - B_{n-1} = 1/(n-1) - 1/n = n/n(n-1) - (n-1)/n(n-1) = 1/n(n-1) > 1/n^2 = A_n - A_{n-1}$$

Hence, $B_n - A_n > B_{n-1} - A_{n-1} \ge 0$, as required. $\square$

Comment: The inductive hypothesis, above, is that $n \ge 2$ and $A_{n-1} \le B_{n-1}$.

Alternative Solution 1: When $n \ge 2$, we have $1/n^2 < 1/n(n-1) = 1/(n-1) - 1/n$, hence

$$1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + ... + 1/n^2 \le$$

$$1 + (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + ... + (1/(n-1) - 1/n) = 2 - 1/n\ . \qquad \square$$

Solution 2: We shall show, in fact, that $r^n + 1/r^n$ is an integer for all natural numbers n. The cases $n = 0$ and $n = 1$ are trivial. Now suppose, inductively, that $n \ge 1$ and that $r^n + 1/r^n$ and $r^{n-1} + 1/r^{n-1}$ are integers. The real number

$$(r^n + 1/r^n)(r + 1/r) = r^{n+1} + 1/r^{n+1} + r^{n-1} - 1/r^{n-1}$$

is an integers, hence $r^{n+1} + 1/r^{n+1}$ is an integer. $\square$

Comment: The inductive hypothesis, here, is that $n \ge 1$ and $r^n + 1/r^n \in \mathbb{Z} \ni r^{n-1} + 1/r_{n-1}$.

Alternative Solution 2: The cases $n = 1$ and $n = 2$ are trivial. When $n \ge 2$, we have

$$(r + r^{-1})^n = r^n + r^{-n} + \left[\binom{n}{1} (r^{n-2} + r^{2-n}) + ... \right]$$

where the last term in the square brackets is $\binom{n}{m}(r + r^{-1})$ or $\binom{n}{m}$ when $n = 2m + 1$ or $n = 2m$, respectively. Assuming, inductively, that $r^k + r^{-k}$ is an integer for all $1 \le k < n$, then the expression in square brackets is an integer, hence $r^n + r^{-n}$ is an integer.

Solution 3: Part (a). The recurrence relation $x_{n+2} + 2x_{n+1} - 8x_n$ has auxilliary quadratic equation $X^2 + 2X - 8 = 0$, which has distinct solutions $X = (-2 \pm \sqrt{36})/2$, in other words, $X = 2$ or $X = -4$. So $x_n = A2^n + B(-4)^n$ for some A and B. We have $-1 = x_0 = A + B$ and $10 = x_1 = 2A - 4B$, hence $A = 1$ and $B = -2$. Therefore, $x_n = 2^n - 2(-4)^n$.

Part (b). We shall show that $(80, -1, 0, -1)$ and (1, -1, -9, -1) are the only solutions for (c, C, D, γ). First suppose there exists a solution where $D = 0$. Putting $n = 0$, we deduce that $C = -1$. Putting $n = 1$, we deduce that $\gamma = -10$. It follows that -10 must be one of the solutions to $X^2 + 2X + c = 0$. By considering the coefficient of X, we deduce that the quadratic equation has one other solution, namely 8. The constant coefficient is the product of the two solutions, $c = 80$. Plainly, $(80, -1, 0, -1)$ is a solution.

Now suppose there exists a solution with $D \ne 0$. The discriminant of the quadratic equation $X^2 + 2X + c = 0$ is $\Delta = 4(1 - c)$. The form of the expression for x_n implies that $\Delta = 0$, in other words, $c = 1$. Then the unique solution to the quadratic equation is $X = -1$. We deduce that $\gamma = -1$. It follows that $x_n = (C + nD)(-1)^n$ for all $n \in \mathbb{N}$. Since $-1 = x_0 = C$ and $10 = x_1 = (C + D)(-1)$, we have $D = -9$. Conversely, it is clear that $(1, -1, -9, -1)$ is a solution.

Solution 4: Part (a). No, because C_3 has 8 vertices with odd degree.

Part (b). Yes. A planar diagram of C_3 was presented in class. (I omit it here because diagrams take a long time to draw in TeX.)

Part (c). This was done in class. (I omit it here because diagrams take a long time.)

Part (d). No, C_4 is non-planar. To see this, suppose otherwise. Since all cycles have length at least 4, we have $e \leq c(n-2)/(c-2)$ where e is the number of edges, n is the number of vertices and $c = 4$. But $n = 16$ and, since every vertex has degree 4, we have $2e = 4.16$ hence $e = 32$. We deduce that $32 \leq 4(16-2)/2 = 28$, which is a contradiction, as required.

Part (e). Yes, because every vertex of the connected graph C_8 has degree 8, which is even.

Part (f). No, by part (d) and the observation that C_8 has a copy of C_4 as a subgraph.

Solution 5: Let H be a disconnected graph with n vertices and a maximal number of edges e. Then H must be isomorphic to the disjoint union of the complete graphs K_a and K_b for some positive integers a and b with $a + b = n$. Then

$$2e = a(a-1) + b(b-1) = a^2 - a + (n-a)^2 - (n-a) = n(n-1) - 2a(n-a) = n(n-1) - 2ab\ .$$

Without loss of generality, $a \leq b$, hence

$$(a-1)(b+1) = ab + a - b - 1 \leq ab - 1 < ab\ .$$

Therefore, noting that a is a positive integer, the maximum value of e is achieved when $a = 1$. In that case, $e = (n-1)(n-2)/2$. We have shown that, among disconnected graphs with n vertices, the maximum possible number of edges is $(n-1)(n-2)/2$ (achieved only when the graph is the disjoint union of K_1 and K_{n-1}). The required conclusion follows. □

Alternative Solution 5: We may assume that $n \geq 3$ and that G has no vertex of degree $n-1$, because otherwise the required conclusion is trivial. Since $n(n-3)/2 < (n-1)(n-2)/2$, some vertex x of G must have degree $n-2$. Let y be the unique vertex of G not adjacent to x. All the vertices distinct from y lie in the same component as x. But those vertices have at most $(n-1)(n-2)/2$ edges between them. So $d(y) \geq 1$ and it follows that y lies in the same component as all the other vertices. □