MATH 527: *Topics in Representation Theory.* Midterm

LJB, 22 March 2016, Bilkent.

Please put your name on every sheet of your manuscript.

Warning: For each question, the length of the solution must be equivalent to, at most, one page in handwriting of the size of this text, with plenty of whitespace. Beyond that length, all excess writing will be ignored.

1: 20% Let K be a field and let C_9 denote the cyclic group of order 9. Up to isomorphism, how many simple KG-modules are there:

(a) when $K = \mathbb{C}$?

(b) when $K = \mathbb{R}$?

(c) when $K = \mathbb{Q}$?

2: 20% Let G be a finite group. Suppose there exists a faithful irreducible $\mathbb{C}G$-character χ.

(a) Show that the centre $Z(G)$ is cyclic.

(b) Let $H \leq G$ and suppose that ${}_H\mathrm{res}_G(\chi)$ is irreducible. Show that $C_G(H) = Z(G)$.

3: 20% The semidihedral group of order 16 has presentation

$$\mathrm{SD}_{16} = \langle a, b : a^8 = b^2 = 1, bab^{-1} = a^3 \rangle \ .$$

Find the ordinary character table of SD_{16}, clearly explaining your method.

4: 20% Let E be the elementary abelian group of order 9. Let $G = Q_8 \ltimes E$ as a semidirect product, where Q_8 acts on E with kernel $Z(Q_8)$. Find the degrees of the irreducible $\mathbb{C}G$-characters and the multiplicity of each degree. (Warning: Avoid routines that would be too long for the examiner to read.)

5: 20% (As Çisil might confirm, this problem has arisen in connection with the study of the trivial-source biset functor.) Let p be a prime and m an integer coprime to p. Let R be the cyclic group with order m, let P be an elementary abelian p-group and let $G = R \ltimes P$ where the action of R on P is such that, writing $\mathbb{F}_p$ for the field of order p, regarding P as an $\mathbb{F}R$-module, then $P \cong \bigoplus_S n_S S$ where S runs over representatives of the isomorphism classes of the simple $\mathbb{F}_p R$-modules and $n_S = 1$ unless S is trivial, in which case $n_S \in \{0, 2\}$. Note that every automorphism α of G fixes P and hence induces an automorphism $\overline{\alpha}$ of R. Show that the group homomorphism $\mathrm{Aut}(G) \ni \alpha \mapsto \overline{\alpha} \in \mathrm{Aut}(R)$ is surjective. (Hint: writing $|P| = p^r$, embed R in $\mathrm{GL}_r(p)$ and consider the normalizer of R in $\mathrm{GL}_r(p)$. Also compare P with the regular $\mathbb{F}_p R$-module.)

Solutions to MATH 527, Midterm 1, Spring 2016.

1:

2:

3:

4:

5: