MATH 325 : *Representation Theory*

MATH 524 : *Group Representations*

Midterm

LJB, 4 April 2018, Bilkent.

Please put your name on every sheet of your manuscript.

Warning: For each question, the length of the solution must be equivalent to, at most, one page in handwriting of the size of this text, with plenty of whitespace. Beyond that length, all excess writing will be ignored.

1: 20% Let K be an algebraically closed field. Up to isomorphism, how many 10-dimensional semisimple algebras over K are there?

2: 20% Let R be a semisimple ring. Show that the center $Z(R)$ is a semisimple ring. Express the number of of isomorphism classes of simple $Z(R)$-modules in terms of $k(R)$, the number of isomorphism classes of simple R-modules.

3: 20% Give an example of a field K and a finite-dimensional algebra A over K such that A not semisimple. For your example, find an ideal J such that J is nilpotent and A/J is semisimple.

4: 20% Let G be a finite group. Let χ_1, ..., χ_k be the irreducible $\mathbb{C}G$-characters. Let g_1, ..., g_k be representatives of the conjugacy classes of G. You may assume the row orthonormality relation $\sum_{\ell=1}^{k} |[g_\ell]| \chi_i(g_\ell^{-1}) \chi_j(g_\ell) = \delta_{i,j} |G|$.

Prove the column orthonormality relation $\sum_{\ell=1}^{k} \chi_\ell(g_i^{-1}) \chi_\ell(g_j) = \delta_{i,j} |C_G(g_i)|$.

5: 20% Let K be a normal subgroup of a finite group G. Let χ be an irreducible $\mathbb{C}G$ character that is not inflated from G/K. Show that, for all $g \in G$, we have

$$\sum_{k \in K} \chi(gk) = 0 = \sum_{k \in K} \chi(kg) \ .$$

(Hint: consider $\sum_k k/|K|$ as an idempotent of $Z(\mathbb{C}G)$.)

Solutions to Midterm.

1: There are exactly 4 isomorphism classes of such algebras because the positive integer solutions to $n_1^2 + ...n_k^2 = 10$ are:

$$1^2 + ... + 1^2 = 1^2 + ... + 1^2 + 2^2 = 1^2 + 1^2 + 2^2 + 2^2 = 1^2 + 3^2 = 10 \, .$$

2: First suppose R is simple as well as semisimple, in other words, $R \cong \mathrm{Mat}_n(\Delta)$ where n is a positive integer and Δ is a division ring. We claim that $Z(R) \cong Z(\Delta)$. Fixing an isomorphism $R \cong \mathrm{Mat}_n(\Delta)$, let $\epsilon_{i,j}$ be the element of R corresponding to the matrix with 1 in the (i,j)-entry and 0 in all the other entries. Let $z \in Z(R)$ and write $z = \sum_{i,j} z_{i,j}\epsilon_{i,j}$ with each $z_{i,j} \in R$. In the case $i \neq j$, a consideration of the equality $\epsilon_{i,i} z \epsilon_{j,j} = z\epsilon_{i,i}\epsilon_{j,j} = 0$ yields $z_{i,j} = 0$. By considering the equality $z\epsilon_{i,j} = \epsilon_{i,j} z$, we see that each $z_{i,i} = z_{j,j}$. Finally, by considering Δ-multiples of the unity element 1_R, we see that each $z_{i,i} \in Z(\Delta)$. The claim is now established.

Generally, write

$$R \cong \bigoplus_{\ell=1}^{k(R)} \mathrm{Mat}_{n_\ell}(\Delta_\ell)$$

as the sum of the Wedderburn components, where each Δ_ℓ is a division ring. By the claim,

$$Z(R) = \bigoplus_{\ell=1}^{k(R)} Z(\Delta_\ell)$$

as a direct sum of fields. In particular, $k(Z(R)) = k(R)$.

3: Let K be any field and let A be a K-module with a basis $\{1, j\}$. We make A become an algebra over K by imposing the multiplication operation whereby 1 is the unity element and $j^2 = 0$. The K-submodule J generated by j is a nilpotent ideal and the quotient A/J is isomorphic to K, which is semisimple.

4: Let X be the $k \times k$ matrix such that the (i,l) entry is $\chi_i(g_\ell)/\sqrt{|C_G(g_\ell)|}$. Bearing in mind that $|G|/|[g_l]| = |C_G(g_\ell)|$ and that $\chi_i(g_\ell^{-1})$ is the complex conjugate of $\chi_i(g_\ell)$, we see that the row orthonormality condition on the character table says precisely that X satisfies the row orthonomality condition in the definition of a unitary matrix, in other words, $X.X^\dagger = 1$. That condition is, of course, equivalent to the column orthonormality condition $X^\dagger.X = 1$, which is precisely the required identity.

5: For any ϕ in the set $\mathrm{Irr}(\mathbb{C}G)$ of irreducible $\mathbb{C}G$-characters, let e_ϕ denote the unity element of the Wedderburn component associated with ϕ. Let $e = \sum_k k/|K|$. Plainly, e is an idempotent of $\mathbb{C}G$. Since $K \trianglelefteq G$, we have $e \in Z(\mathbb{C}G)$. Therefore

$$e = \sum_{\phi \in I} e_\phi$$

for some subset $I \subseteq \mathrm{Irr}(\mathbb{C}G)$.

For any $\phi \in \mathrm{Irr}(\mathbb{C}G)$, we have $ee_\phi = e_\phi$ if and only if e acts as the identity element on a simple $\mathbb{C}G$-module S_ϕ affording ϕ. But each $k \in K$ acts on S_ϕ as a sum of $\phi(1)$ roots of unity.

By the triangle inequality applied to $|K|\phi(1)$ roots of unity, e acts as the identity on S_ϕ if and only if each $k \in K$ acts as the identity on S_ϕ. Therefore, I is the subset of $\mathrm{Irr}(\mathbb{C}G)$ consisting of those ϕ that are inflated from G/K. In particular, $ee_\chi = 0$. But

$$ee_\chi = \frac{\chi(1)}{|G|.|K|} \sum_{f \in G, k \in K} \chi(f^{-1})kf \ .$$

The required equality now follows by evaluating the coefficient of g^{-1}.