MATH 224: Linear Algebra 2

Midterm

19 March 2024, LJB

The duration of the exam is 120 minutes. It is a closed book exam.

You may take the question sheet home.

1: (30 marks.) Consider the linear coding scheme over $\mathbb{F}_2$ with generating matrix

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} .$$

(a) Encode the message words 100, 110, 111.

(b) What is the Hamming matrix H for the coding scheme?

(c) Explain why the received word 10000 must have decoding 000, but 01000 and 00100 do not necessarily have decoding 000.

(d) Write down a decoding table, including the column of syndromes, ensuring that the received words 01000 and 00100 have decoding 000.

(e) Using that decoding table, for the received words 11100, 01110, 00111, write down the syndromes and the decoded words.

2: (30 marks.) Consider the matrix

$$A = \begin{bmatrix} -5 & 3 & 1 \\ -8 & 5 & 2 \\ -7 & 3 & 3 \end{bmatrix}$$

over $\mathbb{C}$. You may assume that the only eigenvalues of A are -1 and 2.

(a) Find an invertible matrix P and a Jordan matrix J such that $A = PJP^{-1}$.

(b) Let $B = (A + I)(A - 2I)$, where I denotes the identity $3{\times}3$ matrix. Without evaluating B or doing any further calculation, find a basis for the kernel of B.

3: (20 marks) Let x be a nonzero vector in $\mathbb{F}_2^3$. How many $3{\times}3$ matrices A over $\mathbb{F}_2$ are there such that $Ax = 0$? (Hint: the set of such matrices A can be regarded as a vector space over $\mathbb{F}_2$.)

4: (20 marks.) Let A be an $n{\times}n$ matrix over an algebraically closed field. Show that A is similar to the transpose matrix A^T. (Recall, the matrices similar to A are the matrices having the form PAP^{-1} where P is invertible.)

Solutions to Midterm

1: Part (a). We have $G100 = 10011$ and $G110 = 11010$ and $G111 = 11100$.

Part (b). We have $H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$.

Part (c). The minimal weight received words for each possible syndrome are as shown.

syndrome	minimal weight words
00	00000
01	01000, 00001
10	00100, 00010
11	10000

So there are exactly 4 possible decoding tables and, for any such table, the 4 received words with decoding 000 must be 00000, 10000, one of 01000 or 00001, one of 00100 or 00010.

Part (d). In the following decoding table, the top row lists the message words and the rightmost column lists the syndromes.

000	001	010	011	100	101	110	111	
00000	00110	01001	01111	10011	10101	11010	11100	00
01000	01110	00001	00111	11011	11101	10010	10100	01
00100	00010	01101	01011	10111	10001	11110	11000	10
10000	10110	11001	11111	00011	00101	01010	01100	11

Part (e). From (d), we obtain the next table.

received	syndrome	decoding
11100	00	111
01110	01	001
00111	01	011

2: The trace of A is $-5 + 5 + 3 = 3 = -1 + 2.2$, so the eigenvalues -1 and 2 must have multiplicities 1 and 2, respectively. Therefore, we can put

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} .$$

Let $v = (x, y, z)$ in $\mathbb{C}^3$. Then v is a -1-eigenvector of A if and only if $(A + I)v = 0$, that is,

$$-4x + 3y + z = -8x + 6y + 2z = -7x + 3y + 4z = 0 ,$$

equivalently, $x = y = z$. So A has -1-eigenvector $f_1 = (1, 1, 1)$. Meanwhile, v is a 2-eigenvector of A if and only if $(A - 2I)v = 0$, that is,

$$-7x + 3y + z = -8x + 3y + 2y = 0 ,$$

equivalently, $v = (x, 2x, x)$. So A has 2-eigenvector $f_2 = (1, 2, 1)$, furthermore, every 2-eigenvector of A is a scalar multiple of f_2. Since the eigenvalue 2 has multiplicity 2, there must exist a generalized 2-eigenvector f_3 such that $(A-2I)f_3 = f_2$. Writing $f_3 = (x, y, z)$, then

$$-7x + 3y + z = 1 \ , \qquad\qquad -8x + 3y + 2z = 2 \ .$$

So we can put $f_3 = (0, 0, 1)$. Taking P to be the coordinate transformation matrix to coordinates with respect to the standard basis from coordinates with respect to the basis $\{f_1, f_2, f_3\}$, also taking J to be the Jordan matrix associated with that basis, we have

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \ , \qquad\qquad J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \ .$$

The construction of P and J ensures that $A = PJP^{-1}$.

Comment: As a direct check, the above matrices P and J satisfy

$$PJP^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 4 & 2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = A \ .$$

Part (b). The subspace $\ker(B) = E_{-1}(A) \oplus E_2(A)$ has basis $\{f_1, f_2\} = \{(1,1,1), (1,2,1)\}$.

3: In the following argument, all the vector spaces are over $\mathbb{F}_2$. Let V be the vector space consisting of the matrices A as specified. Then V is isomorphic to the vector space W of operators α on $\mathbb{F}_2^3$ such that $\alpha(x) = 0$. Let $\mathcal{F} = \{f_1, f_2, f_3\}$ be an ordered basis for $\mathbb{F}_2^3$ with $f_1 = x$. Let X be the vector space consisting of those 3×3 matrices over $\mathbb{F}_2$ whose first column is zero. The condition $\alpha(f_1) = 0$ is precisely the condition that the matrix representing α belongs to X. So W is isomorphic to X. We have $\dim(X) = 6$. So the number of possible A is $|V| = |W| = |X| = 2^6 = 64$.

Comment: As a variant of the argument, we can consider a subspace U of V complementary to $\mathbb{F}_2 x$. Since α is determined by the restriction of α to U, there is an isomorphism between W and the 6-dimensional space of linear maps $\mathbb{F}_2^3 \leftarrow U$. Thus, $\dim(W) = 6$ and, again, the number of possible A is $|V| = |W| = 2^6$.

4: Let F be the scalar field. By the Jordan Normal Form Theorem, $A = PJP^{-1}$ where $P, J \in \mathrm{Mat}_n(F)$ with P invertible and J a Jordan matrix. We have $A^T = QJ^TQ^{-1}$ where $Q = (P^{-1})^T = (P^T)^{-1}$. So we may assume that $A = J$. Writing $A = \mathrm{diag}(J_1, ...)$ where J_1, ... are Jordan blocks, then $A^T = \mathrm{diag}(J_1^T, ...)$. So we may assume that A is a Jordan block. Then $A^T = RAR$ where R is the permutation matrix that reverses the order of the coordinates, we mean the (i, j) entry of R is 1 or 0 depending on whether $i + j = n + 1$ or otherwise, respectively.

Comment: An alternative argument would be to examine the ranks of the matrices having the form $\prod_\lambda (A - \lambda I)^{m_\lambda}$, and similarly for A^T, where λ runs over the eigenvalues of A and the m_λ are natural numbers. We omit the details.

Comment: Two square matrices with the same characteristic polynomial need not be similar. So it is not enough to show that A and A^T have the same characteristic polynomial. Two square matrices with the same characteristic polynomial and the same minimal polynomial again need not be similar.

Actually, it is a nice exercise to show that, for $n \leq 3$, two $n \times n$ matrices over F are similar if and only if they have the same characteristic polynomial and the same minimal polynomial. It is another nice exercise to find a counter-example for any given $n \geq 4$.