MATH 220: Linear Algebra

Midterm

12 March 2024, LJB

The duration of the exam is 120 minutes. It is a closed book exam.

Answers are to be justified by showing the working, explaining the reasoning.

You make take the question sheet home.

1: (10 marks.) Find all the real numbers a, b, c, d such that

$$a + b + c + d = 10 \ , \qquad 2a + 3b + 4c + 5d = 40 \ , \qquad 4a + 5b + 6c + 7d = 60 \ .$$

2: (20 marks.) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \end{bmatrix}$.

(a) Find a basis for the kernel of A. (The kernel is also called the null space.)

(b) Evaluate the rank and nullity of A.

3: Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix}$.

(a) (20 marks.) Find the inverse of A by Gauss–Jordan elimination.

(b) (20 marks.) Find the inverse of A by the method of minors and cofactors.

4: (20 marks.) Which of the following statements are true for all positive integers n, all $n \times n$ real matrices A and all finite subsets $\{s_1, ..., s_m\}$ of $\mathbb{R}^n$? In each case, give a general argument to show why the statement is true, or a single numerical example to show why the statement is false.

(a) If $\{As_1, ..., As_m\}$ is linearly independent, then A is invertible.

(b) If $\{As_1, ..., As_m\}$ spans $\mathbb{R}^n$, then A is invertible.

(c) If A is invertible and $\{s_1, ..., s_m\}$ is linearly independent, then $\{As_1, ..., As_m\}$ is linearly independent.

(d) If A is invertible and $\{s_1, ..., s_m\}$ is a basis for $\mathbb{R}^n$, then $\{As_1, ..., As_m\}$ is a basis for $\mathbb{R}^n$.

5: (10 marks.) Let x be a nonzero vector in $\mathbb{R}^5$. Let V be the vector space consisting of the 5×5 real matrices A such that $Ax = 0$. Evaluate $\dim(V)$.

Solutions to Midterm

1: The augmented matrix is $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 2 & 3 & 4 & 5 & 40 \\ 4 & 5 & 6 & 7 & 60 \end{array}\right]$.

Subtracting multiples of row 1 from the other rows gives $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & 1 & 2 & 3 & 20 \\ 0 & 1 & 2 & 3 & 20 \end{array}\right]$.

Subtracting row 2 from the other rows, we obtain $\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -10 \\ 0 & 1 & 2 & 3 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

The linear system has been reduced to

$$a = c + 2d - 10 \,, \qquad b = -2c - 3d + 20 \,.$$

So the solutions are

$$(a, b, c, d) = (c + 2d - 10, -2c - 3d + 20, c, d)$$

where c and d can be any real numbers.

2: Part (a). By the previous question,

$$\ker(A) = \{(c + 2d. - 2c - 3d, c, d) : c, d \in \mathbb{R}\} = \text{span}\{(1, -2, 1, 0), (2, -3, 0, 1)\}$$

which has basis $\{(1, -2, 1, 0), (2, -3, 0, 1)\}$.

Part (b). By part (a), $\text{null}(A) = 2$. By the rank-nullity formula, $\text{rank}(A) = 4 - \text{null}(A) = 2$.

3: Part (a). We set up the problem as $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right]$.

Subtracting multiples of row 1 from the other two rows, we obtain $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 4 & 8 & -1 & 0 & 1 \end{array}\right]$.

Dividing row 2 by 2 gives $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 4 & 8 & -1 & 0 & 1 \end{array}\right]$.

Subtracting 4 times row 2 from row 3 gives $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 8 & 1 & -2 & 1 \end{array}\right]$.

Dividing row 3 by 8 gives $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/8 & -1/4 & 1/8 \end{array}\right]$.

Subtracting row 3 from row 1 gives $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 7/8 & 1/4 & -1/8 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/8 & -1/4 & 1/8 \end{array}\right]$.

Adding row 2 to row 1 gives $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/8 & 3/4 & -1/8 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/8 & -1/4 & 1/8 \end{array}\right]$.

We obtain $A^{-1} = \frac{1}{8}\begin{bmatrix} 3 & 6 & -1 \\ -4 & 4 & 0 \\ 1 & -2 & 1 \end{bmatrix}$.

Part (b). The matrix of minors for A is $\begin{bmatrix} 9-3 & 9-1 & 3-1 \\ -9-3 & 9-1 & 3+1 \\ -1-1 & 1-1 & 1+1 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 2 \\ -12 & 8 & 4 \\ -2 & 0 & 2 \end{bmatrix}$.

Hence, $\text{adj}(A) = \begin{bmatrix} 6 & 12 & -2 \\ -8 & 8 & 0 \\ 2 & -4 & 2 \end{bmatrix}$.

From the $(1,1)$ entry of the equality $A.\text{adj}(A) = \det(A).I$, we have $\det(A) = 6 + 8 + 2 = 16$. Therefore, $A^{-1} = \text{adj}(A)/16$, and we obtain the same value of A^{-1} as in part (a).

4: Part (a). False, for example, when $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $m = 1$ and $s_1 = (1, 0)$.

Part (b). True. We have $\text{rank}(A) = n$, hence $(A) = 0$ and A is invertible.

Part (c). True. Supposing $\sum_j \lambda_j A s_j = 0$ with each $\lambda_j \in \mathbb{R}$ then, left multiplying by A^{-1}, we deduce that $\sum_j \lambda_j s_j = 0$, so each $\lambda_j = 0$.

Part (d). True. Since $\{s_1, ..., s_m\}$ is a basis for $\mathbb{R}^n$, we have $m = n$. The required conlcusion now follows using part (c).

5: Let $\mathcal{D} = \{d_1, ..., d_5\}$ be a basis for for $\mathbb{R}^5$ such that $d_5 = x$. For all i and j with $1 \le i \le 5$ and $1 \le j \le 4$, there exists a unique 5×5 matrix $A_{i,j}$ sending d_j to d_i and sending all the other elements of $\mathcal{D}$ to 0. The matrices $A_{i,j}$ comprise a basis for V, so $\dim(V) = 20$.