MATH 132, Discrete and Combinatorial Mathematics, Fall 2015

Course specification

Laurence Barker, Bilkent University, version: 29 December 2015.

Course Aims: To supply an introduction to some concepts and techniques associated with discrete mathematical methods in engineering and information technology; in particular, to provide experience of the art of very clear deductive explanation.

Course Description: The terms *combinatorics* and *discrete mathematics* have similar meanings. The former refers to an area of pure mathematics concerned with mathematical objects that do not have very much topological, geometric or algebraic structure. The latter refers to an area of applicable mathematics that rose to prominence with the advent of electronic computers and information technology. Of course, the two cultures overlap considerably and cannot be clearly distinguished from each other.

In the 1950s and 60s, pioneers of computing and computer science found that the established styles of applied mathematics were unsuitable for the new kinds of problem that were appearing. Unlike the classical applied fields such as differential calculus, linear algebra and statistics, the new kind of mathematics was not conducive to formalism, that is to say, methods of calculation based on manipulation of written symbols. Applied mathematicians found that they needed to adopt a conceptual approach which had previously been mostly confined to pure mathematics and some areas of physics.

In discrete mathematics, as opposed to classical applied mathematics, solutions to problems tend to comparatively unsystematic, though certain fundamental ideas do tend to be used quite frequently. For that reason, the study of discrete mathematics depends heavily on the art of *very clear deductive explanation*, which will be emphasized throughout the course.

The course is intended for students who have little or no previous experience of this kind of mathematics. There are no course prerequisites, in fact, proficiency at formal methods of symbolic manipulation will confer no advantage.

We shall be studying three main areas, separate but with some interactions: (1) graph theory; (2) relations and enumerative combinatorics; (3) coding theory.

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Text: R. P. Grimaldi, "Discrete and Combinatorial Mathematics", 5th Ed. (Pearson, 2004).

Some notes will be supplied, on my webpage, for some of the syllabus material.

Classroom hours: All the classes are in room G 236.
Section 1, Monday 15:40 - 16:30, Thursday 13:40 - 15:30.
Section 2, Tuesday 13:40 - 14:30, Thursday 15:40 - 17:30.

Office Hours: in the classroom, G 236, or in my office, room SAZ 129 of Fen A Building.

Section 1, Monday 16:40 - 15:30.
Section 2, Tuesday 14:40 - 15:30.

Office Hours is not just for the stronger students. If you cannot do the easy questions, and if you do not even understand very much of the course material then, (provided you have at least thought about it and have something to talk about), come and see me during office hours. If you think the best grade you can get is a C, then I will help you get that C (and maybe, just maybe, you might surprise yourself and get better than that). I will not be annoyed about your dreadful weakness at this kind of mathematics, because I already know that there are always many students who are dreadfully weak at it; I cannot be of much help to them if they do not come to see me!

You may also come to Office Hours for help with the homework. I will not solve the problem for you, but I can give you guidance on how to do it.

Class Announcements: All students, including any absentees from a class, will be deemed responsible for awareness of announcements made in class.

Revision Aid: Some past exams, with solutions, can be found in discretepastpapers.pdf, on my homepage.

<u>Assessment</u>

Homeworks: The only way to pick up skill at mathematics is through lots of practise.

You may not copy homeworks and you may not do paraphrase rewrites of homeworks by other people. If you break this rule, then you will not catch up in time for the Midterm and Final exams. Just as you cannot learn to swim by watching other people do it, you cannot aquire mathematical skill by just writing out arguments produced by others, not even if you feel that you are "understanding" it as you copy.

You <u>should</u> discuss the homework with each other. In this way, you will teach each other. If you cannot do a homework question, ask another student or ask me during office hours.

Miniproject Essay: A two-page essay on a topic in discrete mathematics. (A short theoretical mathematical essay may consist of, for instance, some definitions, a theorem and a proof. A short procedural mathematical essay might consist of some definitions, a general description of a procedure, and an illustration of the procedure carried out for an example. Do not write a humanities essay, say, "The anthropology of the number 6 from a neostructuralist perspective".) The rules are as follows:

• The topic may be anything in discrete mathematics, but the material must not have been covered during the lectures for this course.
• The discussion must be comprehensible to competent students in the class. It may not assume any knowledge that lies outside the course syllabus. If you make use of concepts beyond the syllabus, you must explain them clearly.
• You must use the notation and terminology of the course.

• It may be typed or handwritten. If handwritten with large handwriting, it may be three pages long. If the length is more than two pages, or three in the case of large handwriting, then the examiner will not read the extra pages.

You may use any textbook or internet source. However, because of the latter two rules above, you will have to adapt the material that you find. It would be a mistake just to copy from another source.

You are advised not to cover anything that is very advanced or very complicated. The essay will be marked according to how clearly it would communicate something with some non-trivial content to other members of the class.

Participation: This will be a mark awarded to the whole section for collective academic behaviour and participation. Asking questions is usually very helpful. All communication should be addressed to the whole class. (Making a distracting noise by murmuring to your neighbour is not proper academic behaviour.)

Principle of marking: In mathematics, marks for written work are not awarded according to guesses about what the student might have had in mind. They are awarded according to *how helpful the written explanation would be to other students in the class.*

Grading percentages:
- Quizzes, participation and homework, 20%,
- Midterm 35%, (Wednesday, 18 November, time to be arranged.)
- Essay project, 5%
- Final, 40%.

Letter Grades: This is done by the "curve method". A grade C requires an understanding of the concepts and reasonable attempts at the easiest exam questions. That fulfills the aim of the course: a competent grasp at an introductory level.

Some of the exercises and exam questions will be quite difficult. It has to be that way, not only for the benefit of the strongest students, but also because, without difficult questions, it would be hard to see the purpose of the art of *very clear deductive explanation.* However, students aiming for a grade C need not worry about being unable to do the more difficult questions.

FZ Grades: These will be awarded to students satisfying at least one of the following conditions: (a) Very poor attendance (less than 50%); (b) very poor Midterm mark (incompetance at most of the easy routine questions); (c) poor attendance and poor Midterm mark.

Attendance: A minimum of 75% attendance is compulsory. Attendance will be measured by random attendance calls and quizzes.

Syllabus

Sylabus Vote: On Thursday 1st October, a vote was taken on whether the last subject area should be (A) coding theory, or (B) generating functions. Only students attending class on that day were eligible to vote. The vote was overwhelmingly in favour of (A).

Week number: Monday date: Subtopics. Section numbers.

1: 7 Sept: (Classes only on Thursday. Classes start 9 Sept.) Discrete methods and information technology. Examples of problems in discrete mathematics. Sketch of the use of mathematical induction, 4.1.

2: 14 Sept: (The class will be taken by Ergün Yalçın, because I will be recovering from a hospital operation.) Recursive definitions and mathematical induction, 4.2. Second order recurrence relations as an application of induction, 10.2.

3: 21 Sept: (No classes, Kurban Bayram.)

4: 28 Sept: (Four hours as makeup for previous week.) Graphs. Sum of degrees formula. Circuits and Trees, 11.1, 11.2. Criteria for existence of Euler paths or Euler circuits, proved by mathematical induction, 11.3.

5: 5 Oct: Euler's characteristic formula for planar graphs, proved by mathematical induction. The non-planarity of the graphs K_5 and $K_{3,3}$, 11.4.

6: 12 Oct: Graph Colouring, 11.5. Second order recurrence relations as an application of induction, 10.2.

7: 19 Oct: Permutations, combinations, the Binomial theorem, 1.2, 1.3, 1.4.

8: 26 Oct: (No class on Thursday.) Inclusion-Exclusion Principle, 8.1.

9: 2 Nov: Sets and correspondences. Functions. Injections, surjections and bijections, 5.1, 5.2, 5.3, 5.6.

10: 9 Nov: (No class on Tuesday.) Relations. Incidence matrices. Reflexive, irreflexive, symmetric, antisymmetric and transitive relations. Enumeration of relations using incidence matrices, 7.1, 7.2.

11: 16 Nov: (Classes only on Thursday.) Midterm on Wednesday, 18 November. Exam postmortem. Partial ordering relations, 7.3.

12: 23 Nov: Hasse diagrams. Chains and antichains, 7.3.

13: 30 Nov: Equivalence relations and Stirling numbers of the Second Kind, 5.3, 7.4.

14: 7 Dec: Coding theory, Hamming metric, 16.5, 16.6. Hamming bound and Gilbert bound, 16.8.

15: 14 Dec: Parity-check and generator matrics, decoding using syndromes and coset leaders, 16.7, 16.8.

16: 21 Dec: (Classes end 24 December.) Review of enumerative combinatorics and coding theory. More practise at exercises and past exam questions.

Midterm Syllabus

The numberings are chapter and section numbers in Grimaldi.

• Mathematical Induction, 4.1, 4.2.

Test: Do you know what the term *inductive assumption* means? When writing out induction arguments, can you state the inductive assumption? (If not, ask me. Many people find this difficult.)

• Introduction to graph theory, 11.1.

Major result: Letting e be the number of edges, then $2e$ is equal to the sum of the degrees.

• Trees, 12.1.

Definition: A tree is a connected graph with no cycles.
Major result: Given a tree with n vertices and e edges, then $e = n - 1$.

• Euler paths, 11.3.

Euler's Path Theorem: Let r be the number of odd-degree vertices of a connected graph G. Then G has an Euler path if and only if $r = 0$ or $r = 2$. Also, G has an Euler circuit if and only if $r = 0$.

Test: Do you know how to find Euler paths for given graphs?

• Planar graphs, 11.4.

Euler's Characteristic Theorem: Given a connected planar graph G with n vertices and e edges, supposing some planar diagram of G has f faces, then $n - e + f = 2$.

Corollary: Given a connected graph G that is not a tree, and an integer c with $c \geq 3$, supposing that every cycle in G has length at least c, then $e \leq c(n-2)/(c-2)$.

• Second order recurrence relations, 10.1, 10.2.

Assuming $a \neq 0$ and $c \neq 0$, then formula for the solutions to $ax_{n+2} + bx_{n+1} + c = 0$ depends on whether or not the quadratic equation $aX^2 + bX + c = 0$ has a repeated solution.

• Binomial coefficients, binomial theorem. Formal and conceptual proofs of Pascal's relation and other relations for binomial coefficients. The number of ways of putting m plain balls in n coloured boxes is $\binom{m+n-1}{m} = \binom{m+n-1}{n-1}$, proved using binary strings. 1.2, 1.3, 1.4.

Test: How many natural number solutions are there to $x_1 + x_2 + x_3 + x_4 = 10$? Note that this is the same as the number of ways of putting 10 plain balls into 4 coloured boxes.

• Functions, injections, surjections, bijections, 5.1, 5.2, beginning of 5.3.

Test: Given finite sets A and B, express, in terms of the sizes $|A|$ and $|B|$,
(a) the number of functions $A \to B$,
(b) the number of injections $A \to B$.

Not examinable in the Final or Midterm: Russell's Paradox, the Halting Problem, graphs on a torus, composition of functions, invertibility of bijections.

Examinable in the Final but not in the Midterm: graph isomorphism, 11.2; surjections and Stirling numbers 5.3. (We have delayed these because they are best explained in connection with equivalence relations.)

See hw132fall15.pdf for 12 past paper questions recommended for midterm practise.

Syllabus for Final Exam

The numberings are chapter and section numbers in Grimaldi.

Note: All of the Midterm syllabus material is also on the Final Syllabus. The questions in the Final exam will primarily be on the following topics, but they may also presuppose knowledge of the Midterm material.

• Composition of functions, inverses of bijections, 5.6.

Remark: Given bijections $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Easy Test: Let $f : X \to Y$ be an injection and let $g : Y \to Z$ be a bijection. Answer each of the following questions by giving a proof or a counter-example. Must $g \circ f$ be an injection? Must $g \circ f$ be a bijection? Must $g \circ f$ be a surjection?

• Relations on a set. Reflexive, anti-reflexive, symmetric, anti-symmetric, transitive relations. (An anti-reflexive relation on a set X is a relation $\sim$ on X such that $x \not\sim x$ for all $x \in X$. The book uses the misnomer "irreflexive" for this property.) Counting relations using incidence matrices, 7.1, 7.2.

• Partial ordering relations. Hasse diagrams. Recall, a **total ordering** on a set x is a partial ordering $\leq$ on X such that $x \leq y$ or $y \leq x$ for all $x, y \in X$. 7.3.

• Equivalence relations, equivalence classes, 7.4.

Theorem: Given an equivalence relation $\equiv$ on a set X, then every element of X belongs to exactly one equivalence class of $\equiv$. In other words, X is the disjoint union of the equivalence classes.

• Isomorphism of partial orderings. Isomorphism of graphs. Recall, the isomorphism relation is an equivalence relation. Roughly covered by 7.3, 7.4, 11.2.

• Surjections, equivalence relations and Stirling numbers (of the second kind), 5.3, 7.4.

Recall, the Stirling number $S(m, n)$ is:

(a) The number of ways of putting m coloured balls into n plain boxes with no box empty.
(b) The number of equivalence relations with n equivalence classes on a set of size m.

The number $n!S(m, n)$ is:

(a) The number of ways of putting m coloured balls into n coloured boxes with no box empty.
(b) The number of surjections from a set of size m to a set of size n.

One way to calculate Stirling numbers is to build up a table using the recurrence relation in the following theorem.

Theorem: For integers $1 \leq n \leq m$ we have $S(m, 1) = 1 = S(m, m)$ and

$$S(m+1, n) = S(m, n-1) + nS(m, n) \ .$$

• General principles of coding theory, the Haming metric, 16.5, 16.6.

Remark: Let C be a code in $\mathbb{Z}_2^n$. Let d be the minimum distance between distinct codewords of C. We can detect s errors of transmission when $s + 1 \leq d$. We can correct t errors of transmission provided $2t + 1 \leq d$.

Remark: Let C and d be as above. If C is a linear code, then d is the minimum weight of a non-zero codeword.

• The generating matrix $G = [I|A]$. Constructing the decoding table. The parity-check matrix $H = [A^T|I]$. Decoding using coset leaders and syndromes. 16.7, 16.8.

Recall, for an encoding scheme $\mathbb{Z}_2^m \to \mathbb{Z}_2^n$ with generating matrix G as above, the process for decoding a received word r is:
(1) Calculate the syndrome Hr^T.

(2) Find the coset leader s with the same syndrome $Hr^T = Hs^T$.
(3) Calculate the codeword $c = r + s$.
(4) Determine the message word w just by reading off the first m digits of c.

The following item, advertised in the weekly syllabus, is not on the Final Exam syllabus because we did not cover it: Week 12 "Chains and antichains". (This was intended as a reference to Dilworth's Theorem, which asserts the equivalence of two definitions of the *width* of a partial ordering on a finite set. Unfortunately, Dilworth's Theorem is not in the textbook but, at the time of writing, proofs can be found in the Wikipedia article called "Dilworth's Theorem". Let us just state it: A **chain** in a poset P is a totally ordered subposet of P. A poset is said to be **discrete** provided it has no non-trivial chains. An **antichain** is a discrete subposet of P. Dilworth's Theorem asserts that, when P is finite, the maximal size of an antichain is equal to the minimal m such that P is a union of m chains. We call m the **width** of P.)