Homeworks

MATH 323, *Algebra 1*, Fall 2016

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Office Hours: Tuesdays, 16:40 - 17:30 following the one-hour class. Usually in the classroom, otherwise in my office, room SA-129 (in the same building as the classroom). For all students, flying easily, struggling desperately or anywhere in-between, this is the time and place to discuss algebra, talk about coursework, or ask me for help with the homeworks.

Homework 1 due Friday 14 October.

Reminder 1: For a positive integer n, we define the **cyclic group** of order n to be the group $C_n = \{1, a, a^2, ..., a^{n-1}\}$ where $a^n = 1$. Writing $\mathbb{Z}/n$ to denote the ring of modulo n integers, we let $(\mathbb{Z}/n)^\times$ denote the group of invertible elements of $\mathbb{Z}/n$. In class, we observed that

$$(\mathbb{Z}/8)^\times = \{1,3,5,7\} \cong V_4 \cong C_2 \times C_2 \ , \qquad (\mathbb{Z}/9)^\times = \{1,2,4,5,7,8\} \cong C_6 \cong C_2 \times C_3 \ .$$

Reminder 2: In class, we found that, up to isomorphism, the groups of order 5 are C_1, C_2, C_3, C_4, V_4, C_5. We have also found two groups with order 6, namely C_6 and S_3. Later, we will prove that there are no other groups with order 6.

1.1: Up to isomorphism, express the groups $(\mathbb{Z}/15)^\times$ and $(\mathbb{Z}/16)^\times$ and $(\mathbb{Z}/17)^\times$ as direct products of cyclic groups.

1.2: Prove the following group-theoretic version of the Chinese Remainder Theorem: given coprime positive integers m and n, then

$$C_{mn} \cong C_m \times C_n \ .$$

1.3: Find, up to isomorphism, all the groups with order 7 or 8 or 9. (Hint: use Lagrange's Theorem.)

Homework 2 due Friday 28 October.

1.1: Find:
(a) the inverse of 3 in $(\mathbb{Z}/7)^\times$,
(b) the inverse of 7 in $(\mathbb{Z}/31)^\times$,
(c) the inverse of 31 in $(\mathbb{Z}/127)^\times$.

2.2: Prove the following converse to the abelian case of Lagrange's Theorem: given a finite abelian group A and a divisor m of $|A|$, then A has a subgroup B with order $|B| = m$.

2.3: Consider the group $\mathbb{Q}$ under addition. For each positive integer n,
(a) How many elements of order n are there in $\mathbb{Q}$?
(b) How many elements of order n are there in the quotient group $\mathbb{Q}/\mathbb{Z}$?
(c) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.

Homework 3 due Tuesday 6th December.

3.1: Let G be a group and X a G-set. Define a relation $=_G$ on X such that, given $x, y \in X$, then $x =_G y$ provided $x = gy$. Show that $=_G$ is an equivalence relation.

3.2: Find the group of rotational symmetries and the group of rigid symmetries of the cube and the octahedron.

3.3: Find those two symmetry groups for the dodecahedron and the icosahedron.

Revision Questions for Final, which are not to be marked.

0: Let G be a simple group with order 660. How many Sylow 11-subgroups does G have?

To do the Final in the file arch323fall14.pdf, the following information will be helpful: given a subset S of a group G, we define the **subgroup generated by** S, denoted $\langle S \rangle$, to be the smallest subgroup of G such that $S \subseteq \langle S \rangle$. When $\langle S \rangle = G$, we call S a **generating set** for G. When G has a finite generating set, we say that G is **finitely generated**. (The additive group of real numbers $(\mathbb{R}, +)$ is an example of an abelian group that is not finitely generated.)

Also recall, a group C is said to be **cyclic** provided C has a generating set with size 1. The infinite cyclic group C_∞ is unique up to isomorphism. The additive group $(\mathbb{Z}, +)$ is an isomorphic copy of C_∞.

Structure Theorem for Finitely Generated Abelian Groups: *Let A be a finitely generated abelian group. Then $A \cong C_{q_1} \times ... \times C_{q_s}$ where each q_i is either ∞ or a power of a prime. Furthermore, the direct product decomposition is unique in that, if $A \cong C_{r_1} \times ... \times C_{r_t}$, then $s = t$ and there is a permutation $\sigma \in S_s$ such that each $r_i = q_{\sigma(i)}$.*