MATH 323, Algebra 1, Fall 2014

Homeworks, Quizzes

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Office Hours would be a good time to ask me for help with the homeworks.

The course textbook is D. S. Dummit and R. M. Foote, "Abstract Algebra", 3rd Edition (Wiley 2004). Some of the questions are taken from there. The solutions are discussed in class.

Homework 1 due Friday 26th September

1.1: Which of the following binary operations are associative?
(a) The operation $*$ on $\mathbb{Z}$ defined by $a * b = a + b + ab$.
(b) The operation $*$ on $\mathbb{Q}$ defined by $a * b = 2(a * b)$
(c) The operation $*$ on $\mathbb{Z} \times \mathbb{Z}$ defined by $(a, \alpha) * (b, \beta) = (a\beta + b\alpha, \alpha\beta)$.

1.2: Show that the multiplication opertion on $\mathbb{Z}/n\mathbb{Z}$ given by $[a]_n \,.\, [b]_n = [ab]_n$ is well-defined.

1.3: For the following a and b express the greatest common divisor (a, b) in the form $(a, b) = xa + yb$ where x and y are integers.
(i) $a = 20$ and $b = 13$.
(ii) $a = 69$ and $b = 372$.

1.4: Let G be a group such that every non-identity element has order 2. Show that G is abelian.

Quiz 1: *Wednesday 24 September.* Find the orders and the inverses of all 12 elements of the group $\mathbb{Z}/12\mathbb{Z}$.

Solution 1: The elements 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11
have orders 1, 12, 6, 4, 3, 12, 2, 12, 3, 4, 6, 12
and inverses 0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, respectively.

Quiz 2: *Wednesday 1 October.* Show that the group isomorphism relation $\cong$ is an equivalence relation.

Solution 2: Let F, G, H be groups. The identity function on F is an isomorphism $F \to F$, so $F \cong F$ and $\cong$ is reflexive. Suppose that $F \cong G$. Let $\theta : F \to G$ be an isomorphism. For $g_1, g_2 \in G$, writing $g_1 = \theta(f_1)$ and $g_2 = \theta(f_2)$, then

$$\theta^{-1}(g_1 g_2) = \theta^{-1}(\theta(f_1 f_2)) = f_1 f_2 = \theta^{-1}(g_1)\theta^{-1}(g_2) \;.$$

So $\theta^{-1} : G \to F$ is an isomorphism. In particular, $G \cong F$. We have shown that $\cong$ is symmetric.

Suppose that $F \cong G$ and $G \cong H$. Let $\theta : F \to G$ and $\phi : G \to H$ be isomorphisms. For $f_1, f_2 \in F$, we have

$$\phi(\theta(f_1 f_2)) = \phi(\theta(f_1)\theta(f_2)) = \phi(\theta(f_1))\phi(\theta(f_2)) \, .$$

So $\phi_\circ\theta : F \to H$ is an isomorphism. In particular, $F \cong H$. We have shown that $\cong$ is transitive.

Quiz 3: *Wednesday 15 October.* Consider the group $D_{10} = \{1, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$. How many subgroups does D_{10} have? Draw a diagram of the subgroup lattice.

Solution 3: In the problem solved immediate before the quiz, it was shown that all the elements of the form a^ib have order 2 and the other four non-identity elements have order 5. By Lagrange's Theorem, all the proper subgroups of D_{10} are cyclic. So the subgroups are: D_{10}, $\langle a\rangle$, $\langle b\rangle$, $\langle ab\rangle$, $\langle a^2b\rangle$, $\langle a^3b\rangle$, $\langle a^4b\rangle$, $\{1\}$. In particular, there are 8 subgroups. [Diagram of subgroup lattice omitted.]

Homework 2 due Friday 17th October

2.1: Let G and H be groups. Show that $G \times H \cong H \times G$.

2.2: Show that $S_4 \not\cong D_{24}$.

2.3: Let k be a positive integer and put $n = 2k$. Let a and b be elements of D_{2n} such that $a^n = b^2 = 1$ and $ba = a^{-1}b$. Show that if $k \geq 2$ then $Z(D_{2n}) = \{1, a^k\}$. Evaluate $Z(D_4)$.

2.4: Find the order of the element $(1, 2)(3, 4, 5)(6, 7, 8, 9, 10)$ of S_{10}.

Solution 2.1: There is an isomorphism $G \times H \ni (g, h) \leftrightarrow (h, g) \in H \times G$.

Solution 2.2: The group D_{24} has an element with order 12, but the maximum order of an element of S_4 is 4.

Solution 2.3: Write $Z = Z(D_{2n})$. Let i be an integer. Then $(a^ib)a(a^ib)^{-1} = a^{-1}$, so $a^ib \notin Z$. We have $ba^ib^{-1} = a^{-1}$, so $a^i \in Z$ if and only if i is divisible by k. Therefore $Z = \{1, a^k\}$. We have $D_4 = \{1, a, b, ab\}$ with $a^2 = b^2 = (ab)^2 = 1$. Therefore $D_4 \cong V_4$, which is abelian. In particular, $Z(D_4) = D_4$.

Solution 2.4: Let $g = (1, 2)(3, 4, 5)(6, 7, 8, 9, 10)$. Since $(1, 2)$ and $(3, 4, 5)$ and $(6, 7, 8, 9, 10)$ mutually commute, $g^n = (1, 2)^n(3, 4, 5)^n(6, 7, 8, 9, 10)^n$ for any integer n. By considering the actions on $\{1, 2\}$ and $\{3, 4, 5\}$ and $\{6, 7, 8, 9, 10\}$, we see that $g^n = 1$ if and only if n is divisible by 2 and 3 and 5. In other words, the order of g is 30.

Homework 3 due Friday 28th November

3.1: *Exercise 4.1.1 page 116:* Let G be a group acting on a set A. Let $a, b \in A$ and $g \in G$ such that $b = ga$. Show that $G_b = gG_ag^{-1}$. Deduce that, if G acts transitively on A, then the kernel of the action is $\bigcap_{g\in G} gG_ag^{-1}$.

3.2: *Exercise 4.1.3 page 116:* Let A be a set and let G be an abelian subgroup of $\mathrm{Sym}(A)$ such that G acts transitively on A. Show that $ga \neq a$ for all $g \in G - \{1\}$ and $a \in A$. Deduce that $|G| = |A|$.

3.3: *Exercise 4.3.13 page 130:* Find, up to isomorphism, all the finite groups that have exactly 2 conjugacy classes.

3.4: *Exercise 4.3.25 page 131:* Let G be the group of invertible 2×2 matrices over $\mathbb{C}$. Consider the subgroup

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{C},\ ac \neq 0 \right\} .$$

Show that every element of G is conjugate to some element of H. (You may assume standard results from linear algebra.)

Solution 3.1: Given $h \in G$ then the condition $h \in G_b$ is equivalent to the condition $hga = ga$, in other words, $g^{-1}hg \in G_a$. The first part is established. The rider holds because, assuming that G acts transitively, then the kernel of the action is the intersection of the stablizers gG_ag^{-1} of the elements ga.

Solution 3.2: By the previous question, the kernel of the action of G on A is $1 = \bigcup_g gG_ag^{-1}$ for any $a \in A$. But A is abelian, so $gG_ag^{-1} = G_a$ for all a and G. Therefore $G_a = 1$, in other words, $ga \neq a$ when $g \neq 1$. By the Orbit-Stabilizer Equation, $|G| = |A|$.

Solution 3.3: Recall that, given a group G and letting g_1, ..., g_k be representatives of the conjugacy classes of G, we have $1 = 1/|C_G(g_1)| + ... + 1/|C_G(g_k)|$.

Plainly, the group C_2 has exactly 2 conjugacy classes. We shall show that, for every group G with exactly 2 conjugacy classes, $G \cong C_2$. Let $a \in G - \{1\}$. Then $1 = 1/|G| + 1/|C_G(a)|$. But $|G| \geq 2 \leq |C_G(a)|$. The only solution is $|G| = 2 = |C_G(a)|$. In particular, $G \cong C_2$.

Solution 3.4: The required conclusion is equivalent to the assertion that every 2×2 matrix over $\mathbb{C}$ has an eigenvector over $\mathbb{C}$.

Quiz 4: *Wednesday 26 November.* Find the conjugacy classes of the group

$$D_{10} = \langle a, b : a^5 = b^2 = 1, bab^{-1} = a^{-1} \rangle = \{1, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\} .$$

Solution 4: Noting that $aba^{-1} = a^2b$, we see that the conjugacy classes are

$$\{1\} , \quad \{a, a^4\} , \quad \{a^2, a^3\} , \quad \{b, ab, a^2b, a^3b, a^4b\} .$$

Quiz 5: *Friday 12 December.* Find the order of the group $\mathrm{GL}_2(5)$.

Solution 5: Let $\{e, f\}$ be an ordered basis for the field $\mathbb{Z}/5/ZZ$. Each element $g \in \mathrm{GL}_2(5)$ is determined by the elements $g(e)$ and $g(f)$. To choose g, there are $5^2 - 1$ choices for $g(e)$, then $5^2 - 5$ choices for $g(f)$. So the number of choices for g is $(5^2 - 1)(5^2 - 5) = 24.20 = 480$.

Homework 4 to be discussed on Friday 26th December.

4.1: Classify, up to isomorphism, the groups with order 12.

4.2: Classify, up to isomorphism, the abelian groups with order 128.

4.3: Let $K \trianglelefteq G$ and let P be a Sylow p-subgroup of K. Show that $G = N_G(P)K$. (The idea, here, is called the **Frattini argument**.)

4.4: Let G be a finite non-trivial p-group.
(a) Show that $1 < Z(G)$.
(b) Show that there exists a strict normal subgroup $N \lhd G$ such that G/N is abelian.

Solution 4.1: We claim that, for a group G with order 12, there are exactly 5 possible isomorphism classes, namely

$$C_{12} \cong C_4 \times C_3 \;, \quad V_4 \times C_3 \cong C_2 \times C_6 \;, \quad C_4 \ltimes C_3 \;, \quad V_4 \ltimes C_3 \cong D_{12} \;, \quad C_3 \ltimes V_4 \cong A_4$$

where all the semidirect products are the unique non-trivial semidirect products of the sepcified form. The case where G is abelian follows immediately from the Structure Theorem for Finite Abelian Groups. Let P and Q be, respectively, a Sylow 2-subgroup and a Sylow 3-subgroup of G. Note that $P \cong C_4$ or $P \cong V_4$. Also, $Q \cong C_3$. Let a be a generator of Q.

Suppose that G is non-abelian and $Q \lhd G$. By the Semidirect Product Recognition Theorem, $G \cong P \ltimes Q$. The only non-trivial automorphism of Q is $a \mapsto a^{-1}$. If $P \cong C_4$, say $P = \langle w \rangle$, then the only way of constructing a non-trivial semidirect product is to impose the relation ${}^w a = a^{-1}$. If $P \cong V_4$ then there are 3 choices of non-trivial automorphism $P \to \mathrm{Aut}(Q)$, but all of them yield the same group up to isomorphism, $P \ltimes Q \cong D_{12}$.

Suppose that Q is not normal. Then, by Sylow's Theorem, the number of Sylow 3-subgroups of G must be 4. So the number of elements of G with order 3 is 8. There are only 4 other elements of G, so $P \lhd G$. Since $\mathrm{Aut}(C_4) \cong C_2$, which has no elements with order 3, we must have $P \cong V_4$. The only non-trivial way in which a can act on the non-trivial elements of P is as a 3-cycle. Therefore, $G \cong A_4$.

Solution 4.2: By the Structure Theorem for Finite Abelian Groups, any abelian group with order 128 is isomorphic to $(\mathbb{Z}/2^{n_1}\mathbb{Z}) \times ... \times (\mathbb{Z}/2^{n_r}\mathbb{Z})$ for some positive integers n_1, ..., n_r such that $n_1 + ... + n_r = 7$. The form is unique upon imposing the condition $n_1 \geq ... \geq n_r$. So the number of isomorphism classes is equal to the number of partitions of 7. We have

$$7 = 6{+}1 = 5{+}2 = 5{+}1{+}1 = 4{+}3 = 4{+}2{+}1 = 4{+}1{+}1{+}1 = 3{+}3{+}1 = 3{+}2{+}2 = 3{+}2{+}1{+}1$$

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Evidently, the number of isomorphism classes is 15.

Solution 4.3: Given $g \in G$ then, by the uniqueness clause of Sylow's Theorem, there exists $k \in K$ such that ${}^g P = {}^k P$. We have $k^{-1} g \in N_G(P)$, hence $g \in K N_G(P) = N_G(P) K$.

Solution 4.4: Part (a). By the Orbit-Stabilizer Equation, every conjugacy class has order a power of p. But $|G|$ is divisible by p. So the number of singleton conjugacy classes is divisible by p. The union of the singleton conjugacy classes is $Z(G)$. We have shown that $|Z(G)|$ is divisible by p.

Part (b). If G is abelian, then the required conclusion is trivial. Suppose that G is not abelian. Invoking part (a) then, by an inductive argument on $|G|$, the quotient $G/Z(G)$ has a strict normal subgroup L such that $(G/Z(G))/L$ is abelian. Letting $H = \{h \in G : hZ(G) \in L\}$, then $H \lhd G$. Also, $H/Z(G) = L$ and, by the Third Isomorphism Theorem, $G/H \cong (G/Z(G))/L$.