MATH 210, Finite and Discrete Mathematics

Homeworks and Quizzes, Spring 2016

Laurence Barker, Mathematics Department, Bilkent University,
version: 21 April 2016.

Office Hours: Wednesdays 08:40 - 09:30 room SA-129 (on the "first" floor of the same building as the classroom, counting in an upwards direction with the floor of the classroom numbered "zero").

Office Hours would be a good time to ask me for help with the homeworks.

Homework 1 due Friday 12 February.

1.1: (a) Let x_0, x_1, ... be an infinite sequence of complex numbers such that

$$x_{n+2} - 7x_{n+1} + 12x_n = 0$$

and $x_0 = 2$ and $x_1 = 7$. Show that $x_n = 3^n + 4^n$ for all n.
(b) Let y_0, y_1, ... be an infinite sequence of complex numbers such that

$$y_{n+2} - 7y_{n+1} + 12y_n = 0 \ .$$

Show that there exist complex numbers A and B such that, for all n, we have $y_n = A3^n + B4^n$.

1.2: The following theorem is called Philip Hall's Marriage Theorem. We shall be able to state it in a less recreational way later in the course. Prove the theorem.

Theorem: Suppose that, at a party, there are finitely many boys and girls and that, for any set S of the boys, there are at least $|S|$ girls g such that at least one of the boys in S knows g. Show that it is possible to monogamously marry all the boys in such a way that every boy is married to a girl who he knows.

Solution 1.1: Let us do part (b) first. FISH to be completed.

Solution 1.2: FISH to be completed.

Quiz 1: *Friday 12 February.* Let n be a positive integer. Show that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \ .$$

One solution: Let A_n and B_n be the left-hand expression and the right-hand expression, respectively. Plainly, $A_1 = B_1$. Now suppose, inductively, that $n \geq 2$ and that $A_{n-1} = B_{n-1}$. We have

$$B_n - B_{n-1} = \frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2 - (n+1)(n-1)}{n(n+1)} = \frac{1}{n(n+1)} = A_n - A_{n-1} \ .$$

Cancelling, we deduce that $A_{n+1} = B_{n+1}$. $\square$

Another solution: We have

$$\frac{1}{1.2} + \frac{1}{2.3} + ... + \frac{1}{n(n+1)} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1} \,. \qquad \square$$

Homework 2, Recurrence relations, due Friday 4 March.

These questions are based on Questions 4, 6, 14, 24 in Section 10.2 of Grimaldi.

Warning: Homework 3 is also due on Friday 4th March because we shall be using the following week for Midterm revision.

2.1: Cars and motorbikes are to be parked in a row of n spaces. The cars are indistinguishable from each other, and each car takes up 2 spaces. The motorbikes are all indistinguishable from each other, and each motorbike takes up 1 space. All n spaces are to be used. How many distinguishable parking arrangements are there? (For example, if $n = 3$, there is one arrangement with 0 cars and 2 arrangements with 1 car, and no other arrangements, hence $1 + 2 = 3$ arrangements in total.)

2.2: Answer the previous question under the change of rules where empty spaces are now allowed.

2.3: Consider an alphabet consisting of 7 numeric characters and k runic characters. For $n \geq 0$, let a_n be the number of words of length n that contain no consecutive runic characters. (That is to say, any two runic characters in the word are separated by at least one numeric character.) Suppose that $a_{n+2} = 7a_{n+1} + 63a_n$ for all $n \geq 0$. What is the value of k?

2.4: Let a_n be the number of ways of covering a $2 \times n$ rectangle using 1×2 rectangles and 2×2 rectangles. Thus, for example, $a_2 = 3$. Find and solve a recurrence relation for a_n.

Homework 3, Graph Theory, due Friday 4 March.

Questions 3.1 and 3.2 are based on Grimaldi 11.3.2 and 11.3.30.

3.1: Consider the connected graphs with 17 edges such that every vertex has degree at least 3. What is the maximum possible number of vertices? (Hint: To complete your proof, you may have to exhibit a graph with the appropriate number of vertices.)

3.2: Ali and Eli attend a party with three other couples. At the party:

- No-one shook hands with his or her spouse.
- No-one shook hands with himself or herself.
- No two people shook hands with each other more than once.

After the party, Eli phoned each of the seven other people to ask how many times he or she had shaken hands. She received seven different answers. How many times did Eli shake hands at the party? How many times did Ali shake hands?

3.3: Let G be a directed multigraph.
(a) A **directed circuit** of G is a circuit of G that traverses every edge in the given direction. A **directed Euler circuit** is a directed circuit that uses every edge exactly once. Show that G has a directed Euler circuit if and only if G is connected and $\mathrm{id}(x) = \mathrm{od}(x)$ for all vertices x of G.
(b) Defining the notions of a **directed path** and a **directed Euler path** similarly, give a similar necessary and sufficient condition for G to have a directed Euler path but no directed Euler circuit.

3.4: For any positive integer n, the n**-cocube** C_n^* is the graph in real n-dimensional space such that: the vertices are the points having the form $z = (z_1, ..., z_n)$ where all of the coordinates z_i are zero except one of them which is 1 or -1; any two vertices z and z' have an edge between them except when $z = z'$ or $z = -z'$. (Thus, $zandz'$ are adjacent if and only if the distance between them is $\sqrt{2}$. As examples, note that C_2^* is the graph of a square and C_3^* is the graph of an octahedron. For which positive integers n does C_n^* have an Euler circuit?

Quiz 2: *Friday 19 February.* Find a general formula for x_n where $x_0 = 2$ and $x_1 = 3i$ (as usual, $i = \sqrt{-1}$) and $2x_{n+2} - 4ix_{n+1} - 2x_n = 0$ for all natural numbers n.

Solution: The quadratic equation $2X^2 - 4iX - 2 = 0$ has repeated solution $X = i$. So $x_n = (C + nD)i^n$ for some C and D. Putting $n = 0$ yields $C = 2$. Putting $n = 1$ yields $C + D = 3$, hence $D = 1$. Therefore $x_n = (2 + n)i^n$ for all n.

Quiz 3: *Friday 26 February.* Does there exist a graph with 100 vertices such that, for all integers m in the range $0 \leq m \leq 99$, there is a vertex with degree m?

Solution: No, such a graph does not exist. Indeed, consider a graph with 100 vertices. Plainly, some vertex has degree 0, then every vertex has degree at most 98. Also, if some vertex has degree 99, then every vertex has degree at least 1.

Past paper revision questions suitable for Midterm 1

These questions can be found in the file discretepastpapers.pdf on my homepage. That file also has solutions and comments on common mistakes made by candidates who took the exam.

- Page 2 (MATH 132, Fall 2015, Midterm). Quesions 2, 3, 5.
- Page 6 (MATH 132, Fall 2015, Midterm Makeup). Quesions 1, 2, 4, 5.
- Page 2 (MATH 210, Spring 2015, Midterm 1). Quesions 1, 2, 3, 4, 5, 6.
- Page 2 (MATH 110, Fall 2014, Midterm). Quesions 1, 2, 3, 4.

Quiz 4: *Wednesday 23 March.* Complete the proof of the following proposition: *Let n be a natural number and m an integer. Given a finite set S of size $|S| = n$, then the number of subsets $T \subseteq S$ with size $|T| = m$ is* $\begin{pmatrix} n \\ m \end{pmatrix}$. Our proof begins as follows: *Let t be the number of such subsets T. The number of ways of arranging m distinct elements of S in order is...*

Solution:

Homework 4, Enumerative Combinatorics, due Wednesday 6 April.

4.1: Let W, X, Y, Z be sets. Let $f : X \leftarrow Y$ be a function. Which two of the following three statements are equivalent to each other. (Give a proof to show that they are equivalent, and give a counter-example to show that the other statement is not equivalent.)
(1) For all functions $g, h : Y \leftarrow Z$ satisfying $f_\circ g = f_\circ h$, we have $g = h$.
(2) For all functions $d, e : W \leftarrow X$ satisfying $g_\circ f = h_\circ f$, we have $g = h$.
(3) The function f is injective.

4.2: How many solutions $(x_1, x_2, x_3, x_4, x_5)$ are there to the equation $x_1+x_2+x_3+x_4+x_5 = 12$ where each x_i is an integer with $x_i \geq 1$?

4.3: Given an integer $n \geq 2$, how many distinguishable ways are there of arranging n mutually distinct objects in a circle if
(a) Two arrangements are the same if one can be obtained from the other by rotation (in other words, the arrangement is determined if we know the left-hand neighbour of each object)?
(b) Two arrangements are the same if one can be obtained from the other by rotation or reflection (in other words, the arrangement is determined if we know the two neighbours of each object, possibly without knowing which neighbour is on the left and which on the right)?

4.4: Consider the graph C_{16} of the 16-dimensional cube. Recall, the vertices of C_{16} are the binary strings of length 16, and two vertices have an edge between then if and only if they differ at exactly one digit. Let x be a vertex of C_{16}. How many vertices y are there such that there is a path of length 6 from x to y?

Equivalent version of the problem: A string of 16 binary digits x is transmitted to us. Suppose we know the received binary string and we also know that, exactly 6 times during transmission, a binary digit was changed. How many possibilities are there for x?

Quiz 5: *Friday 1 April.* Consider the graph represented by a 5×5 square lattice. How many paths of length 8 are there from the bottom left corner to the top right corner?

Bonus part, 3 extra marks: Consider an 8×8 lattice with vertices (x, y) where x, y are integers in the range $1 \leq x \leq 8 \geq y \geq 1$. How many paths of length 8 from $(2, 2)$ to $(6, 6)$ are there if we add edges so that, identifying edges of a surronding square in the evident appropriate ways (depicted in class),
(a) The square becomes a torus? (Both pairs of opposite edges of the square are glued with the same alignment.)
(b) The square becomes a Klein bottle? (Only one of the two pairs of opposite edges are glued with the same alignment.)
(c) The square becomes a real projective plane? (Neither of the pairs of opposite edges are glued with the same alignment.)

Solution:

Quiz 7: *Friday 15 April.* On the set $\{000, 001, 010, 011, 100, 101, 110, 111\}$ of binary strings of length 3, we impose the partial ordering $\leq$ such that $x_1x_2x_3 \leq y_1y_2y_3$ provided, for each $i \in \{1, 2, 3\}$, if $x_i = 1$ then $y_i = 1$. Thus, for example, $010 \leq 011 \leq 111$. Draw a Hasse diagram of this poset.

Exercises suitable for Final Exam preparation

In the Final Exam, all topics in the course are examinable but the focus will be on the topics introduced after Midterm 2: partial orderings, Stirling numbers, coding theory.

The following six questions, and example solutions, can be found in the file discretepastpapers.pdf on my homepage:

• Questions 1, 2, 3 on page 8, • Questions 1, 2, 3 on page 25.

Practise Final

This will not be marked. We shall discuss the solutions in the Special Office Hours on Wednesday 4 May, 08:40 - 10:30.

Question 1: 40% Consider the coding scheme with parity-check matrix

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

(a) Write down the generating matrix G and the codewords for each of the 8 message words.
(b) Without writing down the decoding table, explain why the set of coset leaders consists of: the 6-digit binary string with weight 0, all the 6-digit binary strings with weight 1, one 6-digit binary string with weight 2. How many possible choices of the weight 2 coset leader are there? (Recall, the *coset leaders* are the received words that appear in the first column of the decoding table.)
(c) Without writing down the decoding table, decode the words 000001, 000011, 000111, 001111.
(d) How many errors of transmission can be detected? How many errors of transmission can be corrected? Briefly justify your answers.

Question 2: 20% Suppose we have 16 balls such that each ball is labelled with a binary string of length 4 and no two balls are labelled with the same binary string. How many ways are there of putting the balls into 4 indistinguishable boxes such that:
• every box contains at least one ball labelled with a binary string whose first digit is 0.
• every box contains exactly two balls labelled with binary strings whose first digit is 1.

Question 3: 20% How many isomorphism classes of partial orderings are there on a set with size 4?

Question 4: 20% For sets X and Y, a **correspondence from X to Y** is defined to be a subset of $X \times Y$. Given a correspondence $\sim$ from X to Y and elements $x \in X$ and $y \in Y$, we write $x \sim y$ when $(x, y) \in \sim$.

Now suppose that X and Y are finite and that, given any subset A of X, there are at least $|A|$ elements b of Y for which there exists an element $a \in A$ satisfying $a \sim b$. Show that there exists an injection $f : X \to Y$ such that, for all $x \in X$, we have $x \sim f(x)$. (Hint: first deal with the case where, given any non-empty subset A of X, there are at least $|A| + 1$ elements b of Y for which there exists an $a \in A$ satisfying $a \sim b$.)