Homework

MATH 224, *Linear Algebra 2*, Spring 2024, Laurence Barker

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Homework 1: Review of fundamental theory of vector spaces and linear maps

I do not intend to give solutions to this homework but, if you have queries, do ask.

Exercise 1.A: Let F be a field, let U and V be finite-dimensional vector spaces over F and let α be a linear map $U \to V$. Prove the rank-nullity formula

$$\dim_F(U) = \mathrm{rank}(\alpha) + \mathrm{null}(\alpha)\ .$$

Hint: Extend a basis $\mathcal{A}$ of $\ker(\alpha)$ to a basis $\mathcal{B}$ of U and show that $\alpha(\mathcal{B} - \mathcal{A})$ is a basis for $\mathrm{im}(\alpha)$.

Exercise 1.B: Let X and Y be subspaces of a finite-dimensional vector space V. Show that

$$\dim(X + Y) + \dim(X \cap Y) = \dim(X) + \dim(Y)\ .$$

Hint: Let $\mathcal{A}$ be a basis for $X \cap Y$, extend $\mathcal{A}$ to a basis $\mathcal{X}$ for X and extend $\mathcal{A}$ to a basis $\mathcal{Y}$ for Y. Show that $\mathcal{X} \cup \mathcal{Y}$ is a basis for $X + Y$.

For a vector space V over a field F, an F-linear map $V \to V$ is called an **operator** on V.

Exercise 1.C: Let α be an operator on a finite-dimensional vector space V. Show that the following three conditions are equivalent:

(a) we have $V = \mathrm{im}(\alpha) \oplus \ker(\alpha)$,

(b) we have $V = \mathrm{im}(\alpha) + \ker(\alpha)$,

(c) we have $\mathrm{im}(\alpha) \cap \ker(\alpha) = 0$.

Hint: Use the previous two exercises.

Exercise 1.D: Give an example of a field F, a finite-dimensional F-vector space V and an operator α on V such that the equivalent conditions in Exercise 1.C fail.

Exercise 1.E: Give an example of a field F, an F-vector space V and an operator α on V such that some but not all of the conditions in Exercise 1.C hold.

When we come to the theory of Jordan Normal Form, we shall be considering the following constructions associated with an operator α on a finite-dimensional vector space V over a field F. We define the **Fitting image** of α to be

$$\mathrm{im}^\infty(\alpha) = \bigcap_{m=1}^{\infty} \mathrm{im}(\alpha^m)\ .$$

We define the **Fitting kernel** to be

$$\ker^\infty(\alpha) = \bigcup_{m=1}^{\infty} \ker(\alpha^m)\ .$$

Neither the terminology nor the notation here seems to be standard. Different texts employ different notation.

Exercise 1.F, easy: Show that $\mathrm{im}^\infty(\alpha)$ and $\ker^\infty(\alpha)$ are subspaces of V.

Exercise 1.G, hard: Prove the following theorem. It is named after Hans Fitting, whose doctoral dissertation and death were both in the 1930s.

Fitting's Theorem: (1930s, presumably.) *With the notation above,* $V = \mathrm{im}(\alpha^\infty) \oplus \ker(\alpha^\infty)$.

Hint: Show that, for sufficiently large m, we have $\mathrm{im}^\infty(\alpha) = \mathrm{im}(\alpha^m)$ and $\ker^\infty(\alpha) = \ker(\alpha^m)$.

Homework 2

Question 2.1: The **modulus** of a quaternion $q = t + ix + jy + kz$, with $t, x, y, z \in \mathbb{R}$, is defined by the Pythagorean formula

$$|q| = \sqrt{t^2 + x^2 + y^2 + z^2}\ .$$

As elements of $\mathrm{Mat}_2(\mathbb{C})$, write

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ , \qquad \mathcal{I} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\ , \qquad \mathcal{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\ , \qquad \mathcal{K} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\ .$$

(a) Calculate the determinant of the matrix $tI + x\mathcal{I} + y\mathcal{J} + z\mathcal{K}$.
(b) Using part (a), show that, given $q, q' \in \mathbb{H}$, then $|qq'| = |q|.|q'|$.
(c) Let q be a non-zero quaternion. Again writing $q = t + ix + jy + kz$, give a formula for q^{-1}.

Question 2.2: A field F is said to be **finite** provided the underlying set F is finite, in other words, provided F has only finitely many elements. Show that, if F is finite, then $|F|$ is a power of a prime, in other words, $|F| = p^m$ for some prime p and some positive integer m. (Hint: First show that F contains a copy of the ring $\mathbb{Z}/n$ for some positive integer n. Then make use of the fact that every finite-dimensional vector space has a basis.)

Question 2.3: Suppose F is finite. Consider a system of F-linear equations $Ax = y$. Thus, x and y are column vectors over F and A is a matrix over F. Show that, for each y, the number $|\{x : Ax = y\}|$ is zero or a power of a prime.

Question 2.4: Consider the linear coding scheme with Hamming matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}\ .$$

(a) Write down the generating matrix G for the coding scheme.

(b) Write down a decoding table, including the column of syndromes.

(c) Encode the message words 100, 110, 111.

(d) For the received words 00011, 00111, 01111, 11111, write down the syndromes, then write down the decoded words.

(e) What is the rate of the code?

(f) If a single codeword is transmitted, what is the maximum number of errors of transmission (the maximum number of inversions of binary digits) such that any error can be detected? What is the maximum number of errors of transmission (the maximum number of inversions of binary digits) such that any error can be corrected?

Solutions 2

2.1: Part (a). We have

$$\det(tI + x\mathcal{I} + y\mathcal{J} + z\mathcal{K}) = \begin{vmatrix} t - ix & -y + iz \\ y + iz & t + ix \end{vmatrix} = |t + ix|^2 + |y + iz|^2 = |q|^2 \ .$$

Part (b). The required equality follows from part (a) because $\det(AB) = \det(A)\det(B)$ for 2×2 matrices A and B over $\mathbb{C}$.

Part (c). By the formula for the inverse of a 2×2 matrix with non-zero determinant, if $q \neq 0$, then

$$\begin{bmatrix} t - ix & -y + iz \\ y + iz & t + ix \end{bmatrix}^{-1} = (t^2 + x^2 + y^2 + z^2)^{-1} \begin{bmatrix} t + ix & y + iz \\ -y + iz & t - ix \end{bmatrix}$$

and hence $q^{-1} = (t - ix - jy - kz)/(t^2 + x^2 + y^2 = z^2)$.

2.2: Let K be the subring of F generated by the unity element 1_F. Let p be the smallest positive integer such that $p1_F = 0$. Then K is a copy of the ring $\mathbb{Z}/p$ of modulo p integers. For $x, y \in K$, if $xy = 0$ then $x = 0$ or $y = 0$. Therefore, p is prime and K is a field.

We can regard F as a vector space over K with the same addition operation and with scalar multiplication operation $K \times F \to F$ coming from the multiplication operation of F. Letting m be the dimension of F as a K-vector space, then $|F| = p^m$.

2.3: Let $S = \{x : Ax = y\}$. We may assume that $S \neq \emptyset$. Let $x_0 \in S$. For a column vector x with the appropriate number of coordinates, we have $x \in S$ if and only if $x - x_0$ belongs to the kernel of A. So $|S| = |\ker(A)| = |F|^{\mathrm{null}(A)}$. In Question 2, we showed that $|F|$ is a power of a prime. Therefore, $|S|$ is a power of a prime.

2.4: Part (a). We have $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Part (b). The decoding table is as follows.

000	001	010	011	100	101	110	111	syndrome
00000	00101	01001	01100	10010	10111	11011	11110	00
00001	00100	01000	01101	10011	10110	11010	11111	01
00010	00111	01011	01110	10000	10101	11001	11100	10
00011	00110	01010	01111	10001	10100	11000	11101	11

Part (c). The encodings of 100, 110, 111 are 10010, 11011, 11110, respectively.

Part (d). The received words 00011, 00111, 01111, 11111 have syndromes 11, 10, 11, 01 and decodings 000, 001, 011, 111, respectively

Part (e). The rate is 3/5.

Part (f). The minimum weight of a nonzero codeword is 2. So up to 1 error of transmission can always be detected, and up to 0 errors of transmission can always be corrected.

Homework 3: Exercises on the Midterm examinable material on JNF.

Exercise 3.1: Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} .$$

Find an invertible matrix P and a matrix E in Jordan normal form such that $A = PEP^{-1}$.

Exercise 3.2: Let F be an algebraically closed field of characteristic 0. Let V be a finite-dimensional vector space over F. Let α an operator on V such that $\alpha^n = \mathrm{id}_V$ for some positive integer n. Show that α is diagonalizable.

Exercise 3.3: Let F be a field of prime characteristic p. Let V be a finite-dimensional vector space over F. Let α be an operator on V such that $\alpha^p = \mathrm{id}_V$ and any two eigenvectors of α are F-multiples of each other. Show that $\dim(V) \leq p$.

Exercise 3.4: Let F be a field of prime characteristic p. Let V be an F-vector space with a basis $\{e_i : i \in \mathbb{Z}/p\}$. Let α be the operator on V such that $\alpha(e_i) = e_{i+1}$. Extending to the algebraic closure of F, find the Jordan normal form of α.

Solutions 3

3.1: Let $\{e_1, e_2, e_3\}$ be the standard basis for F^3, where F is the scalar field. Plainly, the unique eigenvalue of A is 1. For any eigenvector $f = xe_1 + ye_2 + ze_3$ of A, we have $x + y + z = x$ and $y + z = y$, hence $y = z = 0$. So, putting $f_1 = (1, 0, 0)$ then, up to scalar multiples, f_1 is the unique eigenvector of A, and the Jordan normal form of A must be

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} .$$

We see f_2 and f_3 such that $\{f_1, f_2, f_3\}$ is a basis for $\mathbb{R}^3$, also $Af_2 = f_2 + f_1$ and $Af_3 = f_3 + f_2$. We can put $f_2 = e_2$. Writing $f_3 = (x, y, z)$ with respect to the standard basis, then

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = f_3 + f_2 = \begin{bmatrix} x \\ y+1 \\ z \end{bmatrix} .$$

Equating coordinates, $x + y + z = x$ and $y + z = y + 1$. So $z = 1$ and $y = -1$. We can put $x = 0$. Then $f_3 = (0, -1, 1)$. The i-th column of p is f_1. So

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} .$$

3.2: Let $J_m(\mu)$ be a Jordan block of α. We must show that $m = 1$. Suppose otherwise. The matrix $J_m(\mu)^n = I_n$ has $(1, 1)$ entry $\mu^n = 1$, so $\mu \neq 0$. The same matrix has $(1, 2)$ entry $n\mu^{n-1} = 0$. But $\mathrm{char}(F) = 0$, so $\mu = 0$. This is a contradiction, as required.

3.3: Let $\beta = \alpha - \mathrm{id}_V$. By the Binomial Theorem, $\beta^p = 0$. So 0 is the unique eigenvalue of β. The eigenvectors of α are precisely the eigenvectors of β. So $(\beta) = 1$. By considering the extension to the algebraic closeure of F, we reduce to the case where F is algebraically closed, whereupon condition $(\beta) = 1$ implies that the Jordan normal form of β is the Jordan block $J_n(0)$, with $n = \dim(V)$. But $J_n(0)^p = 0$, so $n \leq p$.

3.4: We have $\alpha^p = \mathrm{id}_V$. By the Binomial Theorem, $X^p - 1 = (X - 1)^p$ as polynomials over F. So 1 is the unique eigenvalue of α. Let x be an eigenvector and write $x = \sum_i x_i e_i$ with $x_i \in F$. Since $\alpha(x) = x$, the function $i \mapsto x_i$ is constant. Therefore $\dim(E_1(\alpha)) = 1$ and the Jordan normal form of α is $J_p(1)$.