What a group is, according to Plato

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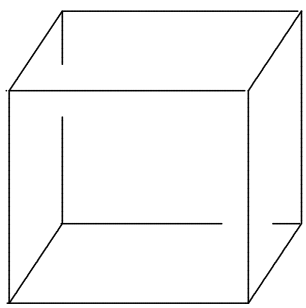
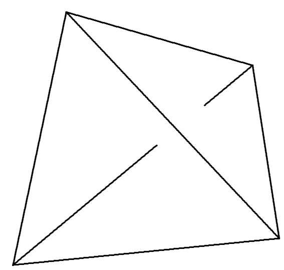
What properties of mathematical objects are we to study? If we were to say that we study their *essence*, then that would sound too mystical. A more scientific sounding term, used especially by the French, is *structure*. We say that we study the *structure* of mathematical objects. Of course, when we say that, what we really mean is that we study their essence.

A cube has exactly 6 faces. A regular tetrahedron has exactly 4 faces. These are structural properties, we mean to say, essential properties. It is not just that some particular sugar-cube in some cafe in Athens happens to have 6 faces. Rather, the Platonic idea of a cube, in heaven, has 6 faces and it follows that, down here in the mundane world, every accurate copy of that idea — realized, say, as dice or sugar-cubes — has 6 faces.





Now let us try something more abstract. Inspecting the left-hand diagram above, we see that a cube has 24 rotational symmetries, as follows. To confirm that the list is complete, note that there are 8 vertices and there are 3 rotations fixing any given vertex; 8 times 3 is 24.

• There are 8 rotations through a third of a full revolution about axes passing through two opposite vertices. Each of these rotations moves 3 of the 4 long diagonals.
• There are 3 rotations through half a full revolution about axes passing through the centres of opposite faces. Each of these moves all 4 of the long diagonals.
• There is 1 rotation through an angle of zero. This operation leaves the cube unmoved. Let us not debate whether this really ought to be called a rotation. We include it for mathematical convenience. The operation moves none of the long diagonals
• There are 6 rotations through half a full revolution about axes through the midpoints of opposite edges. Each of them moves 2 of the long diagonals.
• There are 6 rotations through a quarter of a revolution about axes passing through the centers of opposite faces. Each of them moves all 4 of the long diagonals.

Inspecting the right-hand diagram, we see that a regular tetrahedron has 12 rotational symmetries, as follows. To confirm that the list is complete, observe that there are 4 vertices and there are 3 rotations fixing any given vertex: 4 times 3 is 12.

• There are 8 rotations through a third of a full revolution about axes which pass through a vertex and the centre of a face. Each of these rotations moves 3 of the 4 vertices.
• There are 3 rotations through half a full revolution about axes through midpoints of opposite edges. Each of them moves all 4 of the vertices.
• There is 1 rotation through an angle of zero. This operation moves none of the vertices.

Rigid symmetries are symmetries that can be obtained by rotations, reflections or by combinations of rotations and reflections. The 24 rigid symmetries of the regular tetrahedron are the 12 rotational symmetries above together with the following 12 further symmetries. To confirm the completeness of the list, observe that there are 4 vertices and 6 rigid symmetries fixing any given vertex.

- There are 6 reflections through planes passing through two vertices and the midpoint of an edge. Each of these reflections moves 2 of the vertices.
- There are 6 rigid symmetries moving all 4 of the vertices.

A group is something which expresses symmetry. One can speak of the group of rotational symmetries of a cube. One can speak of the group of rigid symmetries of a regular tetrahedron. Those two groups are essentially the same. Or, to use more scientific sounding terminology, the two groups have the same structure. The trick to seeing this is to identify the four long diagonals of the cube with the four vertices of the tetrahedron. This group has a standard name. It is called S_4. Thus, S_4 is the name of an abstract idea which can become manifest in various guises, for instance, as the rotational symmetries of the cube, or as the rigid symmetries of a regular tetrahedron.