Genotypes of irreducible representations of finite p-groups

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Abstract

For any characteristic zero coefficient field, an irreducible representation of a finite p-group can be assigned a Roquette p-group, called the genotype. This has already been done by Bouc and Kronstein in the special cases $\mathbb{Q}$ and $\mathbb{C}$. A genetic invariant of an irrep is invariant under group isomorphism, change of coefficient field, Galois conjugation, and under suitable inductions from subquotients. It turns out that the genetic invariants are precisely the invariants of the genotype. We shall examine relationships between some genetic invariants and the genotype. As an application, we shall count Galois conjugacy classes of certain kinds of irreps.

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1 Introduction and conclusions

We shall be concerned with $\mathbb{K}G$-irreps, that is to say, irreducible representations of G over $\mathbb{K}$, where G is a finite p-group, p is a prime, and $\mathbb{K}$ is a field with characteristic zero. Of course, in the study of the irreps of a finite p-group over a field, there is scant loss of generality in assuming that the field has characteristic zero. Roquette [14] showed that every normal abelian subgroup of G is cyclic if and only if G is one of the following groups: the cyclic group C_{p^m} with $m \geq 0$; the quaternion group Q_{2^m} with $m \geq 3$; the dihedral group D_{2^m} with $m \geq 4$; the semidihedral group SD_{2^m} with $m \geq 4$. When these two equivalent conditions hold, we call G a **Roquette p-group**. This paper is concerned with a reduction technique whereby the study of $\mathbb{K}G$-irreps reduces to the case where G is Roquette.

The reduction technique originates in Witt [16] and Roquette [14]. Our main sources are: Kronstein [12], Iida–Yamanda [11] for complex irreps of p-groups; tom Dieck [7, Section III.5] for real irreps of finite nilpotent groups; Bouc [2], [3], [4] for rational irreps of p-groups; Hambleton–Taylor–Williams [10], Hambleton–Taylor [9], for rational irreps of hyperelementary groups.

We shall be taking advantage of the generality of our scenario. In the final section, we shall unify some enumerative results of tom Dieck [7] and Bouc [4] concerning Galois conjugacy classes of rational, real and complex irreps.

Consider a $\mathbb{K}G$-irrep ψ. In a moment, we shall define a Roquette p-group $\mathrm{Typ}(\psi)$, which we shall call the **genotype** of ψ. We shall explain how $\mathrm{Typ}(\psi)$ determines — and is determined by — many other invariants of ψ.

Let us agree on some terminology. When no ambiguity can arise, we may neglect to distinguish between characters, modules and representations. For a $\mathbb{K}G$-rep μ, we write $\mathbb{Q}[\mu]$ for the field generated over $\mathbb{Q}$ by the values of the character μ. We write $\mathrm{End}_{\mathbb{K}G}(\mu)$ to denote the endomorphism algebra of the of the module μ. We write $\mathrm{Ker}(\mu)$ to denote the kernel of the representation μ as a group homomorphism from G. When μ is irreducible, the Wedderburn component of $\mathbb{K}G$ associated with μ is the Wedderburn component that is not annihilated by the representation μ as an algebra homomorphism from $\mathbb{K}G$.

Given subgroups $K \trianglelefteq H \leq G$, then the subquotient H/K of G is said to be **strict** provided $H < G$ or $1 < K$. We understand induction $\mathrm{ind}_{H/K}^G$ to be the composite of induction ind_H^G preceded by inflation $\mathrm{inf}_{H/K}^H$. An easy application of Clifford theory shows that, if some $\mathbb{K}G$-irrep is not induced from a strict subquotient, then G is Roquette. Therefore, any $\mathbb{K}$-irrep of a finite p-group is induced from a Roquette subquotient. For example, the faithful $\mathbb{C}D_8$-irrep ψ_0 is induced from a faithful $\mathbb{C}C_4$-irrep ϕ_0. But this observation, in its own, does not yield a very powerful reduction technique. The $\mathbb{C}$-irreps ψ_0 and ϕ_0 differ in some important respects, for instance, $\mathbb{Q}[\psi_0] = \mathbb{Q}$ whereas $\mathbb{Q}[\phi_0] = \mathbb{Q}[i]$.

Since $\mathbb{K}$ has characteristic zero, we can equally well understand deflation $\mathrm{def}_{H/K}^H$ to be passage to the K-fixed points or as passage to the K-cofixed points. We understand restriction $\mathrm{res}_{H/K}^H$ to be the composite of deflation $\mathrm{def}_{H/K}^H$ preceded by restriction res_H^G. A $\mathbb{K}G$-irrep ψ is said to be **tightly induced** from a $\mathbb{K}H/K$-irrep ϕ provided $\psi = \mathrm{ind}_{H/K}^G(\phi)$ and no Galois conjugate of ϕ occurs in the $\mathbb{K}H/K$-rep $\mathrm{res}_{H/K}^G(\psi) - \phi$. This is equivalent to the condition that, regarding ϕ as a $\mathbb{K}H$-irrep by inflation, then $\psi = \mathrm{ind}_H^G(\phi)$ and no Galois conjugate of ϕ occurs in the $\mathbb{K}H$-rep $\mathrm{res}_H^G(\psi) - \phi$. So, when discussing tight induction, the inflations and deflations are trivial formalities, and we may safely regard $\mathbb{K}H/K$-reps as $\mathbb{K}H$-reps by inflation.

Theorem 1.1. (Genotype Theorem) *Given a $\mathbb{K}G$-irrep ψ, then there exists a Roquette subquotient H/K such that ψ is tightly induced from a faithful $\mathbb{K}H/K$-irrep ϕ. For any such subquotient H/K, the $\mathbb{K}H/K$-irrep ϕ is unique. Given another such subquotient H'/K', then $H/K \cong H'/K'$.*

We call H/K a **genetic subquotient** for ψ, and we call ϕ the **germ** of ψ at H/K. We define the **genotype** of ψ, denoted $\mathrm{Typ}(\psi)$, to be H/K regarded as an abstract group, well-defined only up to isomorphism. The existence of such subquotients H/K, in the case $\mathbb{K} = \mathbb{Q}$, is implicit in Witt [16], explicit in Roquette [14]. The uniqueness, in the case $\mathbb{K} = \mathbb{Q}$, is due to Bouc [2]. Via Lemma 3.2, we see that the existence and uniqueness, in the case $\mathbb{K} = \mathbb{C}$, is due to Kronstein [12].

In Section 4, we shall prove the Genotype Theorem 1.1 indirectly by invoking the Field-Changing Theorem 3.5, which says that the genetic theory is independent of the field $\mathbb{K}$. As a matter of fact, the theory really is independent of $\mathbb{K}$, and there is no need to reduce to a previously established special case. A direct proof of the Genotype Theorem will materialize from some characterizations of $\mathrm{Typ}(\psi)$ in Section 5.

Given a $\mathbb{K}G$-irrep ψ, then there exists a unique $\mathbb{Q}G$-irrep $\psi_{\mathbb{Q}}$ such that ψ occurs in the $\mathbb{K}G$-rep $\mathbb{K}\psi_{\mathbb{Q}} = \mathbb{K} \otimes_{\mathbb{Q}} \psi_{\mathbb{Q}}$. For a field $\mathbb{L}$ with characteristic zero and an $\mathbb{L}G$-irrep ψ', we say that ψ and ψ' are **quasiconjugate** provided $\psi_{\mathbb{Q}} = \psi'_{\mathbb{Q}}$. We write $\psi_{\mathbb{L}}$ to denote an arbitrarily chosen $\mathbb{L}G$-irrep that is quasiconjugate to ψ. In Section 2, as a little illustrative application of the genetic reduction technique, we shall show that, for irreps of finite p-groups over an arbitrary field with characteristic zero, the notion of Galois conjugacy is well-defined and well-behaved. Corollary 2.6 says that two $\mathbb{K}G$-irreps are quasiconjugate if and only if they are Galois conjugate.

Consider a formal invariant $\mathcal{I}$ defined for all irreps of all finite p-groups over all fields with characteristic zero. We call $\mathcal{I}$ a **quasiconjugacy invariant** provided $\mathcal{I}(\psi) = \mathcal{I}(\psi')$ for all characteristic zero fields $\mathbb{L}$ and all $\mathbb{L}G$-irreps ψ' that are quasiconjugate to ψ. If $\mathcal{I}$ is a quasiconjugacy invariant then, in particular, it is a Galois conjugacy invariant, and $\mathcal{I}(\psi_{\mathbb{L}})$ is well-defined, independently of the choice of $\psi_{\mathbb{L}}$. We call $\mathcal{I}$ a **global invariant** provided $I(\psi) = I(\psi^{\&})$ whenever some group isomorphism $G \to G^{\&}$ sends the $\mathbb{K}G$-irrep ψ of G to the $\mathbb{K}G^{\&}$-irrep $\psi^{\&}$ of $G^{\&}$. For instance, $\psi_{\mathbb{Q}}$ is a quasiconjugacy invariant but not a global invariant, while the degree $\psi(1)$ a global invariant but not a quasiconjugacy invariant.

We call $\mathcal{I}$ a **tight induction invariant** provided $\mathcal{I}(\psi) = \mathcal{I}(\phi)$ for all subquotients H/K of G and all $\mathbb{K}H/K$-irreps ϕ such that ψ tightly induced from ϕ. We call $\mathcal{I}$ a **genetic invariant** when $\mathcal{I}$ is a tight quasiconjugacy global invariant, in other words, $\mathcal{I}$ is preserved by tight induction, Galois conjugacy, change of field, and group isomorphism.

Despite the apparent strength of the defining conditions, many interesting invariants of ψ are genetic invariants. See the list at the end of Section 2. A $\mathbb{C}G$-irrep that is quasiconjugate to ψ is called a **vertex** of ψ. The number of vertices, denoted $v(\psi)$, is called the **order** of ψ. In Section 5, we shall see that that $v(\psi)$ is a genetic invariant. Another genetic invariant is the set of vertices $\mathrm{Vtx}(\psi)$, regarded as a permutation set for a suitable Galois group. Yet another genetic invariant is the **vertex field** $\mathbb{V}(\psi)$, which is the field generated over $\mathbb{Q}$ by the character values of a vertex. We shall also see that the genotype $\mathrm{Typ}(\psi)$ is a genetic invariant. In fact, Corollary 5.9 asserts that the genetic invariants of ψ are precisely the invariants of $\mathrm{Typ}(\psi)$.

How can $\mathrm{Typ}(\psi)$ be ascertained from easily calculated genetic invariants such as the order $v(\psi)$ and the vertex set $\mathrm{Vtx}(\psi)$ and the vertex field $\mathbb{V}(\psi)$? How can $\mathrm{Typ}(\psi)$ be used to ascertain less tractable genetic invariants such as the minimal splitting fields? We shall respond to these questions in Section 5. Employing a medical analogy: the patient has red eyes and long teeth, therefore the patient has genotype V_{666}, and therefore the patient is allergic to sunlight. Or, arguing from information in the next paragraph: if $v(\psi) = 2$ and the Frobenius–Schur indicator of ψ is positive, then ψ has genotype D_{16}, hence the unique minimal splitting field for ψ is $\mathbb{Q}[\sqrt{2}]$.

Some examples: the genotype $\mathrm{Typ}(\psi)$ is the trivial group C_1 if and only if ψ is the trivial character; $\mathrm{Typ}(\psi) = C_2$ if and only if ψ is affordable over $\mathbb{Q}$ and non-trivial; $\mathrm{Typ}(\psi) = D_{2^m}$ with $m \geq 4$ if and only if $\psi_{\mathbb{C}}$ is affordable over $\mathbb{R}$ but not over $\mathbb{Q}$, in which case m is determined by the order $v(\psi) = 2^{m-3}$.

2 Galois conjugacy of irreps of p-groups

One starting-point for the genetic theory is the following weak expression of ideas in Witt [16], Roquette [14]. (Although, as we shall explain at the end of this paper, the starting point for the work was actually Tornehave [15].) We are working with the finite p-group G because we have nothing of novel significance to say about arbitrary finite groups. The remark can be quickly obtained by ignoring most of the proof of Lemma 4.2 below.

Remark 2.1. *Any $\mathbb{K}$-irrep of G can be expressed in the form $\mathrm{ind}_{H/K}^{G}(\phi)$ where $K \trianglelefteq H \leq G$ and the subquotient H/K is a Roquette p-group and ϕ is a faithful $\mathbb{K}H/K$-irrep which is not induced from any proper subgroup of H/K.*

In the notation of the remark, the subquotient H/K need not be unique up to isomorphism. When $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, examples of the non-uniqueness of H/K abound. When $\mathbb{K} = \mathbb{Q}$, an example of the non-uniqueness of H/K is supplied by the group $C_4 * D_{16}$. Here, the smash product identifies the two central subgroups with order 2. Incidently, the group $C_4 * D_{16}$ was exhibited by Bouc [2, 7.7] as a counter-example to another assertion. Routine calculations show that, for the unique faithful $\mathbb{Q}C_4 * D_{16}$-irrep, one choice of H/K has the form $(C_4 \times C_2)/C_2 \cong C_4$ and another choice of H/K has the form $(C_8 \times C_2)/C_2 \cong C_8$.

Although the remark yields only a crude version of the genetic reduction technique, we shall be applying it, in this section, to prove the following theorem. Since the theorem is fundamental, classical in style and not very hard to obtain, one presumes that it is well-known, but the author has been unable to locate it in the literature. (Incidently, the author does not know whether it holds for all hyperelementary groups. A negative or absent answer might present an inconvenience to the generalization of the genetic theory to hyperelementary groups.)

We throw some terminology. Given a $\mathbb{K}G$-irrep and a subfield $\mathbb{J} \leq \mathbb{K}$, we define the $\mathbb{J}G$-irrep **containing** ψ to be the unique $\mathbb{J}G$-irrep ψ' such that ψ occurs in the $\mathbb{K}$-linear extension $\mathbb{K}\psi'$.

For a positive integer n, we write $\mathbb{Q}_n$ to denote the field generated over $\mathbb{Q}$ by primitive n-th roots of unity. We call $\mathbb{Q}_n$ the **cyclotomic field** for **exponent** n. By a **subcyclotomic field**, we mean a subfield of a cyclotomic field. Since the Galois group $\mathrm{Aut}(\mathbb{Q}_n) = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ is abelian, any subcyclotomic field is a Galois extension of $\mathbb{Q}$. Observe that, for any $\mathbb{K}$-irrep ψ, the field $\mathbb{Q}[\psi]$ is subcyclotomic.

Theorem 2.2. *Let $\mathbb{L}$ be an extension field of $\mathbb{K}$, let ψ be a $\mathbb{K}$-irrep, and let ψ_1, ..., ψ_v be the $\mathbb{L}G$-irreps contained in the $\mathbb{L}$-linear extension $\mathbb{L}\psi = \mathbb{L} \otimes_{\mathbb{K}} \psi$. Then:*

(1) *Given j, then ψ is the unique $\mathbb{K}G$-irrep containing ψ_j.*

(2) *There exists a positive integer $m_{\mathbb{K}}^{\mathbb{L}}(\psi)$ such that $\mathbb{L}\psi = m_{\mathbb{K}}^{\mathbb{L}}(\psi)(\psi_1 + ... + \psi_v)$.*

(3) *The field $\mathbb{Q}[\psi_j] = \mathbb{Q}[\psi_1]$ is a Galois extension of $\mathbb{Q}[\psi]$.*

(4) $\mathrm{Gal}(\mathbb{Q}[\psi_1]/\mathbb{Q}[\psi])$ *acts freely and transitively on ψ_1, ..., ψ_v. The action is such that an element α of the Galois group sends ψ_j to ψ_k when ${}^{\alpha}(\psi_j(g)) = \psi_k(g)$ for all $g \in G$.*

Part (1) is obvious. Part (2) is an immediate implication of part (4). By a comment above, the subcyclotomic field $\mathbb{Q}[\psi_j]$ is a Galois extension of the subcyclotomic field $\mathbb{Q}[\psi]$. The equality $\mathbb{Q}[\psi_j] = \mathbb{Q}[\psi_1]$ is implied by part (4). When we have proved part (4), we shall have proved the whole theorem.

If the extension $\mathbb{L}/\mathbb{K}$ is Galois then, as explained in Curtis–Reiner [5, 7.18, 7.19], $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ acts transitively on ψ_1, ..., ψ_v. Any element of $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ restricts to an element of $\mathrm{Gal}(\mathbb{Q}[\psi_1]/\mathbb{Q}[\psi])$. A straightforward argument now establishes the theorem in the case where $\mathbb{L}/\mathbb{K}$ is Galois.

Replacing $\mathbb{L}/\mathbb{K}$ by $\mathbb{K}/\mathbb{Q}$, we see that the theorem implies the following proposition.

Proposition 2.3. *Let ψ be a $\mathbb{K}G$-irrep, and let α be an automorphism of a field containing $\mathbb{Q}[\psi]$. Then there exists a $\mathbb{K}G$-irrep ${}^{\alpha}\psi$ such that $({}^{\alpha}\psi)(g) = {}^{\alpha}(\psi(g))$ for all $g \in G$.*

Let us show that, conversely, the proposition implies the theorem. Assuming the proposition, it is easy to deduce that $\mathrm{Gal}(\mathbb{Q}[\psi_1]/\mathbb{Q}[\psi])$ acts freely on ψ_1, ..., ψ_v. It remains only to show that the action is transitive. Let $\mathbb{J}$ be an extension field of $\mathbb{L}$ such that $\mathbb{J}/\mathbb{K}$ is Galois. Let ${}_1\psi$ and ${}_j\psi$ be $\mathbb{J}G$-irreps contained in ψ_1 and ψ_j, respectively. Since the theorem holds for the Galois extension $\mathbb{J}/\mathbb{K}$, there exists an element $\alpha \in \mathrm{Gal}(\mathbb{J}/\mathbb{K})$ such that ${}^{\alpha}_{1}\psi = {}_j\psi$. Then ${}^{\alpha}\psi_1 = \psi_j$ and ${}^{\alpha}\psi = \psi$. Also α restricts to an element of $\mathrm{Gal}(\mathbb{Q}[\psi_1]/\mathbb{Q}[\psi])$. The transitivity of the action is established. We have deduced part (4) of the theorem. In fact, we have shown that the proposition and the theorem are equivalent to each other.

By the remark, proof of the theorem and the proposition reduces to the case where G is Roquette. We must recall the classification of the Roquette p-groups. First, let us recall the members of a slightly different class of extremal p-groups. The following groups are precisely the p-groups with a self-centralizing cyclic maximal subgroup. See, for instance, Aschbacher [1, 23.4]. For $m \geq 3$, the **modular group** with order p^m is defined to be

$$\mathrm{Mod}_{p^m} = \langle a, c : a^{p^{m-1}} = c^p = 1\,,\ cac^{-1} = a^{p^{m-2}+1} \rangle.$$

Still with $m \geq 3$, the **quaternion group** with order 2^m is

$$Q_{2^m} = \langle a, x : a^{2^{m-1}} = 1\,,\ x^2 = a^{2^{m-2}}\,,\ xax^{-1} = a^{-1} \rangle.$$

Again with $m \geq 3$, the **dihedral group** with order 2^m is

$$D_{2^m} = \langle a, b : a^{2^{m-1}} = b^2 = 1\,,\ bab^{-1} = a^{-1} \rangle.$$

For $m \geq 4$, the **semidihedral group** with order 2^m is

$$\mathrm{SD}_{2^m} = \langle a, d : a^{2^{m-1}} = d^2 = 1\,,\ dad^{-1} = a^{2^{m-2}-1} \rangle.$$

We shall refer to these presentations as the **standard presentations**. The only coincidence in the list is $\mathrm{Mod}_8 \cong D_8$. Where the presentations make sense for smaller values of m, the resulting groups are abelian.

Suppose that G is a non-abelian Roquette p-group and let A be a maximal normal cyclic subgroup of G. Let $A \trianglelefteq K \trianglelefteq G$ such that K/A is a cyclic subgroup of $Z(G/A)$. If K/A is contained in the kernel of the action of G/A on A, then K is a normal abelian subgroup of G, hence $K = A$ by the hypotheses on G and A. We deduce that G/A acts freely on A. In other words, A is self-centralizing in G. Hence, via the technical lemma [1, 23.5], we recover the following well-known result of Roquette.

Theorem 2.4. (Roquette's Classification Theorem.) *The Roquette p-groups are precisely the following groups.*

(a) *The cyclic group C_{p^m} where $m \geq 0$.*

(b) *The quaternion group Q_{2^m} where $m \geq 3$.*

(c) *The dihedral group D_{2^m} where $m \geq 4$.*

(d) *The semidihedral group SD_{2^m} where $m \geq 4$.*

It is worth sketching the content of the invoked technical lemma because we shall later be needing some notation concerning automorphisms of cyclic 2-groups. (Besides, there is some charm in the connection between the classical number theory behind Theorem 2.4 and the algebraic number theory in Section 5.) Let v be a power of 2 with $v \geq 2$. The group $\mathrm{Aut}(C_{4v}) \cong (\mathbb{Z}/4v)^\times \cong C_2 \times C_v$ has precisely three involutions, namely the elements b, c, d which act on a generator a of C_{4v} by

$$b : a \mapsto a^{-1}, \qquad c : a \mapsto a^{2v+1}, \qquad d : a \mapsto a^{2v-1}.$$

Any odd square integer is congruent to 1 modulo 8. So b and d cannot have a square root in $\mathrm{Aut}(C_{4v})$. Therefore c belongs to every non-trivial subgroup of $\mathrm{Aut}(C_{4v})$ except for $\langle b \rangle$ and $\langle d \rangle$. Now suppose that G is a 2-group with a self-centralizing normal cyclic subgroup $A = \langle a \rangle$ with index $|G : A| \geq 4$. The inequality implies that $A \cong C_{4v}$ with $v \geq 2$, and moreover, the image of G/A in $\mathrm{Aut}(A)$ must own the involution $c : a \mapsto a^{2v+1}$. Abusing notation, the subgroup $\langle c \rangle$ of $\mathrm{Aut}(A)$ lifts to a normal subgroup $\langle a, c \rangle \cong \mathrm{Mod}_{8v}$ of G. But Mod_{8v} has a characteristic subgroup $\langle a^{2v}, c \rangle \cong V_4$. We deduce that G is not Roquette. The rest of the proof of Theorem 2.4 is straightforward.

Given $H \leq G$, a $\mathbb{K}H$-irrep ϕ and a $\mathbb{K}G$-irrep ψ, then ϕ occurs in $\mathrm{res}^G_H(\psi)$ if and only if ψ occurs in $\mathrm{ind}^G_H(\phi)$. When these two equivalent conditions hold, we say that ψ and ϕ **overlap**. The following observation is an easy consequence of Clifford's Theorem.

Lemma 2.5. *Suppose that G has a self-centralizing normal cyclic subgroup A.*

(1) *Given a $\mathbb{K}G$-irrep ψ overlapping with a $\mathbb{K}A$-irrep ξ, then ψ is faithful if and only if ξ is faithful.*

(2) *Given a faithful $\mathbb{K}G$-irrep ψ overlapping with a faithful $\mathbb{K}A$-irrep ξ, then ψ is an integer multiple of $\mathrm{ind}^G_A(\eta)$. Furthermore, ψ is absolutely irreducible if and only if ξ is absolutely irreducible, in which case, the integer multiple is unity.*

(3) *The condition that ψ and ξ overlap characterizes a bijective correspondence between the faithful $\mathbb{K}G$-irreps ψ and the G-conjugacy classes of faithful $\mathbb{K}A$-irreps ξ.*

Since the Roquette p-groups satisfy the hypothesis of Lemma 2.5, we deduce that any Roquette p-group has a faithful $\mathbb{K}$-irrep.

We let $n(G)$ denote the exponent of G. Brauer's Splitting Theorem asserts that the cyclotomic field $\mathbb{Q}_{n(G)}$ splits for G, that is to say, every $\mathbb{Q}_{n(G)}G$-irrep is absolutely irreducible.

Lemma 2.6. *Suppose that G is Roquette. Then*

(1) *The automorphism group $\mathrm{Aut}(G)$ acts transitively on the faithful $\mathbb{K}G$-irreps.*

(2) *The Galois group $\mathrm{Aut}(\mathbb{Q}_{n(G)}) = \mathrm{Gal}(\mathbb{Q}_{n(G)}/\mathbb{Q})$ acts transitively on the faithful $\mathbb{K}G$-irreps. The action is such that $({}^\alpha\psi)(g) = {}^\alpha(\psi(g))$ where $g \in G$ and ψ is a faithful $\mathbb{K}G$-irrep.*

Proof. Write $n = n(G)$. First suppose that G is cyclic. Then $n = |G|$. Part(1) is clear in this case. There is a triple of commuting isomorphisms between the groups $\mathrm{Aut}(G)$ and $(\mathbb{Z}/n)^\times$ and $\mathrm{Aut}(\mathbb{Q}_n)$ such that, given elements $\aleph$ and & and α, respectively, then $\aleph \leftrightarrow \& \leftrightarrow \alpha$ provided $\aleph(g^{\&}) = g$ and $\alpha(\omega) = \omega^{\&}$ where $g \in G$ and ω is an n-th root of unity. Then ${}^\alpha(\psi(g)) = ({}^\aleph\psi)(g)$. Thus, the specified action of $\mathrm{Aut}(\mathbb{Q}_n)$ on the faithful $\mathbb{K}G$-irreps coincides with the action via the isomorphism $\mathrm{Aut}(\mathbb{Q}_n) \cong \mathrm{Aut}(G)$. Part (2) is now clear in the case where G is cyclic.

Now suppose that G is non-cyclic. The classification of the Roquette p-groups implies that $p = 2$ and G is dihedral, semidihedral or quaternion. So there exists a cyclic maximal subgroup A and an element $y \in G - A$ such that either $y^2 = 1$ or y^2 is the unique involution in A. Any automorphism $\aleph$ of A must fix y^2, so $\aleph$ can be extended to an automorphism @ of G such that

@ fixes y. We have already seen that $\mathrm{Aut}(A)$ acts transitively on the faithful $\mathbb{K}A$-irreps. In view of the bijective correspondence in Lemma 2.5, $\mathrm{Aut}(A)$ acts transitively on the faithful $\mathbb{K}G$-irreps via the monomorphism $\mathrm{Aut}(A) \ni \aleph \mapsto @ \in \mathrm{Aut}(G)$. Part (1) follows perforce. Now suppose that $\aleph$ and α are corresponding elements of $\mathrm{Aut}(A)$ and $\mathrm{Aut}(\mathbb{Q}_n)$. By part (2) of Lemma 2.5 together with the formula for induction of characters, the faithful $\mathbb{K}G$-irreps vanish off A. So ${}^{\alpha}(\psi(g)) = ({}^{@}\psi)(g)$ for all $g \in G$. As before, part (2) follows. □

By the same argument, the conclusions of the lemma also hold for the modular p-groups.

We can now complete the proof of Theorem 2.2 and Proposition 2.3. Above, we showed that the proposition implies the theorem, and we also explained how the proposition reduces to the case where G is Roquette. But that case of the proposition is weaker than part (2) of Lemma 2.6. The theorem and the proposition are now proved.

Corollary 2.7. *Let ψ be a $\mathbb{K}G$-irrep. Let $\mathbb{J}$ be a Galois extension of $\mathbb{Q}[\psi]$. Then $\mathrm{Gal}(\mathbb{J}/\mathbb{Q})$ acts transitively on the on the $\mathbb{K}G$-irreps that are quasiconjugate to ψ. If $\mathbb{J}$ owns primitive $n(G)$-th roots of unity, then two $\mathbb{K}G$-irreps ψ_1 and ψ_2 lie in the same $\mathrm{Gal}(\mathbb{J}/\mathbb{Q})$-conjugacy class if and only if ψ_1 and ψ_2 are quasiconjugate.*

Proof. This follows from Theorem 2.2 by replacing $\mathbb{L}/\mathbb{K}$ with $\mathbb{K}/\mathbb{Q}$. □

When ψ and ψ' satisfy the equivalent conditions in the latest corollary, we say that ψ and ψ' are **Galois conjugate**. Thus, we may speak unambiguously of the Galois conjugates of a given $\mathbb{K}G$-irrep; there is no need to specify the Galois extension and there is no need for the Galois automorphisms to stabilize $\mathbb{K}$ nor even to be defined on $\mathbb{K}$. We can now express part (4) of Lemma 2.6 more succinctly.

Corollary 2.8. *If G is Roquette, then the faithful $\mathbb{K}G$-irreps comprise a single Galois conjugacy class.*

Keeping in mind the above features of Galois conjugacy, we see that the following invariants of a $\mathbb{K}G$-irrep ψ are quasiconjugacy global invariants. Let $\mathbb{J}$ be any field with characteristic zero. In some of the items below, it may seem that we have proliferated notation unnecessarily, and that it would be simpler to present only the case $\mathbb{K} = \mathbb{J}$. However, there is a distinction to be made: the invariants are associated with the field $\mathbb{J}$, whereas the given irrep ψ has coefficient field $\mathbb{K}$. In the applications in Section 5, we shall be mostly concerned with the cases $\mathbb{J} = \mathbb{Q}$ and $\mathbb{J} = \mathbb{R}$, but the given irrep ψ will still have coefficients in arbitrary $\mathbb{K}$. Recall that $\psi_{\mathbb{J}}$ denotes a $\mathbb{J}G$-irrep that is quasiconjugate to ψ. For the first item in the list, we let $\mathbb{L}$ be any field extension of $\mathbb{J}$.

• **The $\mathbb{L}/\mathbb{J}$-relative Schur index $m_{\mathbb{J}}^{\mathbb{L}}(\psi)$ and the $\mathbb{L}/\mathbb{J}$-relative order $v_{\mathbb{J}}^{\mathbb{L}}(\psi)$.** We define them to be the positive integers m and v, respectively, such that the $\mathbb{L}$-linear extension of ψ can be written in the form $\mathbb{L}\psi = m(\psi_1 + ... + \psi_v)$ where ψ_1, ..., ψ_v are mutually distinct $\mathbb{L}G$-irreps. Theorem 2.2 tells us that each ψ_j is a Galois conjugate of $\psi_{\mathbb{L}}$. Schilling's Theorem 5.9 tells us that $m_{\mathbb{J}}^{\mathbb{L}}(\psi) \leq 2$.

• **The endomorphism ring $\mathrm{End}_{\mathbb{J}G}(\psi_{\mathbb{J}})$.** Strictly speaking, the invariant here is the isomorphism class of $\mathrm{End}_{\mathbb{J}G}(\psi_{\mathbb{J}})$ as a $\mathbb{J}$-algebra.

• **The class of minimal splitting fields for $\psi_{\mathbb{J}}$.** Still letting $\mathbb{L}$ be an extension field of $\mathbb{J}$, the $\mathbb{L}$-irrep $\psi_{\mathbb{L}}$ is absolutely irreducible if and only if $\mathbb{L}$ is a splitting field for $\mathrm{End}_{\mathbb{J}G}(\psi_{\mathbb{J}})$, or equivalently, $\mathbb{L}$ is a splitting field for the Wedderburn component of $\mathbb{J}G$ associated with $\psi_{\mathbb{J}}$.

When these equivalent conditions hold, $\mathbb{L}$ is said to be a **splitting field** for ψ. If furthermore, the degree $|\mathbb{I} : \mathbb{J}|$ is minimal, then $\mathbb{L}$ is said to be a **minimal splitting field** for ψ.

Let $\mathbb{M}$ be a splitting field for $\psi_{\mathbb{J}}$. In the next two quasiconjugacy global invariants, the stated properties of $m_{\mathbb{J}}(\psi)$ and $v_{\mathbb{J}}(\psi)$ are well-known and can be found in Curtis–Reiner [5, Section 74].

• **The $\mathbb{J}$-relative Schur index $m_{\mathbb{J}}(\psi)$ and the $\mathbb{J}$-relative order $v_{\mathbb{J}}(\psi)$.** Defined as $m_{\mathbb{J}}(\psi) = m_{\mathbb{J}}^{\mathbb{M}}(\psi)$ and $v_{\mathbb{J}}(\psi) = v_{\mathbb{J}}^{\mathbb{M}}(\psi)$, they are independent of the choice of $\mathbb{M}$. We mention that, if $\mathbb{M}$ is a minimal splitting field for $\psi_{\mathbb{J}}$, then its degree over $\mathbb{J}$ is $m_{\mathbb{J}}(\psi)v_{\mathbb{J}}(\psi) = |\mathbb{M} : \mathbb{J}|$.

• **The $\mathbb{J}$-relative vertex field $\mathbb{V}_{\mathbb{J}}(\psi)$.** This invariant is an isomorphism class of extension fields of $\mathbb{J}$. It has three equivalent definitions: firstly, $\mathbb{V}_{\mathbb{J}}(\psi) = \mathbb{J}[\psi_{\mathbb{M}}]$; secondly, $\mathbb{V}_{\mathbb{J}}(\psi)$ is the centre of the division ring $\mathrm{End}_{\mathbb{J}G}(\psi_{\mathbb{J}})$; thirdly, $\mathbb{V}_{\mathbb{J}}(\psi)$ is the centre of the Wedderburn component of $\mathbb{J}G$ associated with $\psi_{\mathbb{J}}$. We mention that $v_{\mathbb{J}}(\psi) = |\mathbb{V}_{\mathbb{J}}(\psi) : \mathbb{J}|$. In other words, $m_{\mathbb{J}}(\psi) = |\mathbb{M} : \mathbb{V}_{\mathbb{J}}(\psi)|$ when the splitting field $\mathbb{M}$ is minimal. Also, $m_{\mathbb{J}}(\psi)$ is the square root of the dimension of $\mathrm{End}_{\mathbb{J}G}(\psi_{\mathbb{J}})$ over $\mathbb{V}_{\mathbb{J}}(\psi)$.

Our reason for ploughing through this systematic notation is that, in Section 5, we shall show that the above invariants are not merely quasiconjugacy global invariants. They are also tight induction invariants. That is to say, they are genetic invariants. This is a compelling vindication of the proposed notion of tight induction. Also, as a speculative motive for considering the invariants in such generality, let us suggest the possibility of a technique whereby assertions pertaining to arbitrary $\mathbb{K}$ may be demonstrated by first dealing with one of the extremal cases $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{C}$, then establishing a passage for field extensions with prime degree, and then arguing by induction on the length of an abelian Galois group.

However, to characterize the genotype of a given irrep, we shall only be making use of the cases $\mathbb{J} = \mathbb{Q}$ and $\mathbb{J} = \mathbb{R}$. Let us list the genetic invariants that will be of applicable significance in Section 5. Some of them are special cases of the above.

• The endomorphism ring $\mathrm{End}_{\mathbb{Q}G}(\psi_{\mathbb{Q}})$, which is a ring well-defined up to isomorphism.

• The class of minimal splitting fields for $\psi_{\mathbb{Q}}$.

• The vertex field $\mathbb{V}(\psi) = \mathbb{Q}[\psi_{\mathbb{C}}]$. Besides the three equivalent definitions above, another characterization of $\mathbb{V}(\psi)$ will be given in Proposition 5.10 (and this fourth equivalent definition supplies a rationale for the terminology.)

• The **exponent** $n(\psi)$, which we define to be the minimal positive integer such that $\mathbb{Q}_{n(\psi)}$ is a splitting field for $\psi_{\mathbb{Q}}$.

• The **Fein field** of ψ, which we define to be the unique subfield $\mathrm{Fein}(\psi) \leq \mathbb{Q}_{n(\psi)}$ such that $\mathrm{Fein}(\psi)$ is a minimal splitting field for $\psi_{\mathbb{Q}}$. The existence and uniqueness of $\mathrm{Fein}(\psi)$ will be proved in Theorem 5.7. (The existence can fail for arbitrary finite groups.)

• The Schur index $m(\psi) = m_{\mathbb{Q}}(\psi)$ and the order $v(\psi) = v_{\mathbb{Q}}(\psi)$. We mention that $m(\psi)$ is the multiplicity of $\psi_{\mathbb{C}}$ and $v(\psi)$ is the number of Galois conjugates of $\psi_{\mathbb{C}}$. Also,

$$2 \geq m(\psi) = |\mathrm{Fein}(\psi) : \mathbb{V}(\psi)| = \sqrt{\dim_{\mathbb{V}(\psi)}(\mathrm{End}_{\mathbb{Q}G}(\psi_{\mathbb{Q}}))}\,.$$

• The vertex set $\mathrm{Vtx}(\psi)$, which we define to be the transitive $\mathrm{Aut}(\mathbb{V}(\psi))$-set consisting of the $\mathbb{C}G$-irreps that are quasiconjugate to ψ. We sometimes call these $\mathbb{C}G$-irreps the **vertices** of ψ.

Actually, the invariant here is the isomorphism class of $\mathrm{Vtx}(\psi)$ as an $\mathrm{Aut}(\mathbb{V}(\psi))$-set. Putting $v = v(\psi)$ and letting ψ_1, ..., ψ_v be the vertices of ψ, then the $\mathbb{V}(\psi)\,G$-irreps contained in ψ can be enumerated as ψ'_1, ..., ψ'_v in such a way that the $\mathbb{V}(\psi)$-linear extension of ψ decomposes as $\mathbb{V}(\psi)\,\psi = \psi'_1 + ... + \psi'_v$ and the $\mathbb{C}$-linear extension of each ψ'_j decomposes as $\mathbb{C}\psi'_j = m(\psi)\psi_j$. Thus, $\mathrm{Aut}(\mathbb{V}(\psi))$ permutes the $\mathbb{V}(\psi)$-irreps ψ'_j just as it permutes the vertices ψ_j. Note that

$$v(\psi) = |\mathbb{V}(\psi) : \mathbb{Q}| = |\mathrm{Vtx}(\psi)|.$$

(Another rationale for the terminology now becomes apparent.) We point out that, given any field extension $\mathbb{I}$ of $\mathbb{V}(\psi)$, then any automorphism of $\mathbb{I}$ restricts to an automorphism of $\mathbb{V}(\psi)$, hence $\mathrm{Vtx}(\psi)$ becomes an $\mathrm{Aut}(\mathbb{I})$-set.

• The endomorphism algebra $\Delta(\psi) = \mathrm{End}_{\mathbb{R}G}(\psi_\mathbb{R})$ is called the **Frobenius–Schur type** of ψ. Understanding $\Delta(\psi)$ to be well-defined only up to ring isomorphism, then there are only three possible values, namely $\mathbb{R}$ and $\mathbb{C}$ and $\mathbb{H}$. The respective values of the pair $(m_\mathbb{R}(\psi), v_\mathbb{R}(\psi))$ are $(1, 1)$ and $(1, 2)$ and $(2, 1)$. If ψ is given as a $\mathbb{K}G$-character $G \to \mathbb{K}$, then a practical way to determine $\Delta(\psi)$ is to make use of the **Frobenius–Schur** indicator, which is defined to be the integer

$$\mathrm{fs}(\psi) = \frac{1}{|G|} \sum_{g \in G} \psi(g^2).$$

Recall that $\Delta(\psi)$ is $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ depending on whether $\mathrm{fs}(\psi_\mathbb{C}) = 1$ or $\mathrm{fs}(\psi_\mathbb{C}) = 0$ or $\mathrm{fs}(\psi_\mathbb{C}) = -1$, respectively. Also, $\mathrm{fs}(\psi) = m_\mathbb{K}(\psi) v_\mathbb{K}(\psi) \mathrm{fs}(\psi_\mathbb{C})$. Therefore $\Delta(\psi)$ is $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ depending on whether $\mathrm{fs}(\psi) > 0$ or $\mathrm{fs}(\psi) = 0$ or $\mathrm{fs}(\psi) < 0$, respectively. The genetic invariance of $\Delta(\psi)$ is implicit in Yamanda–Iida [18, 5.2].

3 Tight induction

Let us repeat the most important definition in this paper. Consider a subgroup $H \leq G$, a $\mathbb{K}G$-irrep ψ and a $\mathbb{K}H$-irrep ϕ such that ψ is induced from ϕ. When no Galois conjugate of ϕ occurs in $\mathrm{res}^G_H(\psi) - \phi$, we say that ψ is **tightly induced** from ϕ and, abusing notation, we also say that the induction $\psi = \mathrm{ind}^G_H(\phi)$ is **tight**. As we noted in Section 1, the definition extends in the evident way to induction from subquotients. The tightness condition can usefully be divided into two parts, as indicated in the next two lemmas.

Lemma 3.1. (Shallow Lemma) *Given $H \leq G$, and $\mathbb{K}G$-irrep ψ induced from a $\mathbb{K}H$-irrep ϕ, then the following conditions are equivalent:*

(a) *The multiplicity of ϕ in $\mathrm{res}^G_H(\psi)$ is 1.*

(b) *The division rings $\mathrm{End}_{\mathbb{K}H}(\phi)$ and $\mathrm{End}_{\mathbb{K}G}(\psi)$ have the same $\mathbb{K}$-dimension.*

(c) *As $\mathbb{K}$-algebras, $\mathrm{End}_{\mathbb{K}H}(\phi)$ and $\mathrm{End}_{\mathbb{K}G}(\psi)$ are isomorphic.*

Proof. As $\mathbb{K}$-vector spaces, we embed ϕ in ψ via the identifications $\phi = 1 \otimes \phi$ and $\psi = \bigoplus_{gH \subseteq G} g \otimes \phi$. We embed the $\mathbb{K}$-algebra $\mathcal{D} = \mathrm{End}_{\mathbb{K}H}(\phi)$ in the $\mathbb{K}$-algebra $\mathcal{E} = \mathrm{End}_{\mathbb{K}G}(\psi)$ by letting $\mathcal{D}$ kill the module $\theta = \sum_{gH \subseteq G - H} g \otimes \phi$. The relative trace map $\mathrm{tr}^G_H : \mathrm{End}_{\mathbb{K}H}(\psi) \to \mathcal{E}$ restricts to an $\mathbb{K}$-algebra monomorphism $\nu : \mathcal{D} \to \mathcal{E}$. So conditions (b) and (c) are both equivalent to the condition that ν is a $\mathbb{K}$-algebra isomorphism.

Suppose that (a) holds. Then any $\mathbb{K}H$-endomorphism of ψ restricts to a $\mathbb{K}H$-endomorphism of ϕ. In particular, any element $\epsilon \in \mathcal{E}$ restricts to an element $\mu(\epsilon) \in \mathcal{D}$. We have defined a

$\mathbb{K}$-algebra map $\mu : \mathcal{E} \to \mathcal{D}$. From the constructions, we see that $\mu\nu$ is the identity map on $\mathcal{D}$. So μ is surjective. But $\mathcal{D}$ is a division ring, so μ is injective. Hence μ and ν are mutually inverse $\mathbb{K}$-algebra isomorphisms. We have deduced (b) and (c).

Now suppose that (a) fails. Let ϕ' be a $\mathbb{K}H$-submodule of θ such that $\phi' \cong \phi$. Let β be a $\mathbb{K}H$-endomorphism of ψ such that β kills θ and β restricts to a $\mathbb{K}H$-isomorphism $\phi \to \phi'$. Let $\gamma = \mathrm{tr}_H^G(\beta)$. Then β and γ have the same action on ϕ. In particular, γ restricts to an isomorphism $\phi \to \phi'$. On the other hand, any element $\delta \in \mathcal{D}$ has the same action on ϕ as $\nu(\delta)$. In particular, $\nu(\delta)$ restricts to a $\mathbb{K}H$-automorphism of ϕ. Therefore $\gamma \in \mathcal{E} - \nu(\mathcal{D})$ and ν is not surjective. We have deduced that (b) and (c) fail. □

Lemma 3.2. (Narrow Lemma) *Given $H \leq G$, and $\mathbb{K}G$-irrep ψ induced from a $\mathbb{K}H$-irrep ϕ, then the following conditions are equivalent:*

(a) *No distinct Galois conjugate of ϕ occurs in $\mathrm{res}_H^G(\psi)$.*

(b) *The condition $\psi' = \mathrm{ind}_H^G(\phi')$ describes a bijective correspondence between the Galois conjugates ψ' of ψ and the Galois conjugates ϕ' of ϕ.*

(c) *We have $\mathbb{Q}[\phi] = \mathbb{Q}[\psi]$.*

Proof. The equivalence of (a) and (b) is clear by Frobenius Reciprocity. By the standard formula for the values of an induced character, $\mathbb{Q}[\phi] \geq \mathbb{Q}[\psi]$. The fields $\mathbb{Q}[\phi]$ and $\mathbb{Q}[\psi]$ are subcyclotomic, so the field extension $\mathbb{Q}[\phi]/\mathbb{Q}[\psi]$ is Galois. Conditions (b) and (c) are both equivalent to the condition that no Galois automorphism moves ϕ and fixes ψ. □

When the equivalent conditions in Lemma 3.1 hold, we say that ψ is **shallowly** induced from ϕ. When the equivalent conditions in Lemma 3.2 hold, we say that ψ is **narrowly** induced from ϕ. The induction $\psi = \mathrm{ind}_H^G(\phi)$ is tight if and only if it is shallow and narrow. In the special case $\mathbb{K} = \mathbb{Q}$, the narrowness condition is vacuous: an induction of rational irreps $\psi = \mathrm{ind}_H^G(\phi)$ is tight if and only if $\mathrm{End}_{\mathbb{Q}G}(\psi) \cong \mathrm{End}_{\mathbb{Q}H}(\phi)$. The definition of tight induction in the case $\mathbb{K} = \mathbb{Q}$ is due to Witt [16]. At the other extreme, when $\mathbb{K}$ is algebraically closed, the shallowness condition is vacuous: an induction of complex irreps $\psi = \mathrm{ind}_H^G(\phi)$ is tight if and only if $\mathbb{Q}[\psi] = \mathbb{Q}[\phi]$. The definition of tight induction in the case $\mathbb{K} = \mathbb{C}$ is due to Kronstein and, independently, to Iida-Yamanda [11].

Remark 3.3. *Let $H \leq L \leq G$. Let ϕ and $\theta = \mathrm{ind}_H^L(\phi)$ and $\psi = \mathrm{ind}_L^G(\theta)$ be $\mathbb{K}$-irreps of H and L and G, respectively. If any two of the inductions $\theta = \mathrm{ind}_H^L(\phi)$ and $\psi = \mathrm{ind}_L^G(\theta)$ and $\psi = \mathrm{ind}_H^G(\phi)$ are shallow, then all three are shallow. If any two of the inductions are narrow, then all three are narrow. If any two of them are tight, then all three are tight.*

The remark is obvious. It tells us, in particular, that tight induction is transitive. In fact, given a $\mathbb{K}G$-irrep, then there is a G-poset whose elements are the pairs (H, ϕ) such that $H \leq G$ and ϕ is a $\mathbb{K}H$-irrep from which ψ is tightly induced. The partial ordering is such that $(H, \phi) \leq (L, \theta)$ provided $H \leq L$ and θ is induced from ϕ (whereupon, by the remark, θ is tightly induced from ϕ). The Genotype Theorem 1.1 (proved in the next section) implies that the minimal elements of the G-poset are the pairs (H, ϕ) such that $H/\mathrm{Ker}(\phi)$ is Roquette.

Theorem 3.4. *Let $H \leq G$ and let ψ be a $\mathbb{K}G$-irrep induced from a $\mathbb{K}H$-irrep ϕ. Let $\mathbb{J}$ be a subfield of $\mathbb{K}$. Let $\mathbb{L}$ be a field extension of $\mathbb{K}$. Then the following conditions are equivalent:*

(a) *The $\mathbb{J}G$-irrep containing ψ is tightly induced from the $\mathbb{J}H$-irrep containing ϕ.*

(b) *ψ is tightly induced from ϕ.*

(c) *There is a bijective correspondence between the $\mathbb{L}G$-irreps ψ' contained in ψ and the $\mathbb{L}H$-irreps ϕ' contained in ϕ. The correspondence is characterized by the condition that ψ' is tightly induced from ϕ'.*

Proof. When extending the coefficient field for finite-dimensional modules, the extension of the hom-space is the hom-space of the extensions. So the $\mathbb{J}G$-irrep ψ'' containing ψ must overlap with $\mathbb{J}H$-irrep ϕ'' containing ϕ. By Theorem 2.2,

$$\mathbb{K}\psi'' = m_{\mathbb{J}}^{\mathbb{K}}(\psi) \sum_{\alpha \in \mathrm{Gal}(\mathbb{Q}[\psi]/\mathbb{Q}[\psi''])} {}^{\alpha}\psi \, , \qquad \mathbb{K}\phi'' = m_{\mathbb{J}}^{\mathbb{K}}(\phi) \sum_{\beta \in \mathrm{Gal}(\mathbb{Q}[\phi]/\mathbb{Q}[\phi''])} {}^{\beta}\phi \, .$$

Suppose that (b) holds. Then $\mathbb{Q}[\psi] = \mathbb{Q}[\phi]$. Since ϕ'' occurs in $\mathrm{res}_H^G(\psi'')$, since ${}^{\beta}\psi$ is the unique Galois conjugate of ψ overlapping with ${}^{\beta}\phi$, and since ${}^{\beta}\phi$ occurs only once in the restriction of ${}^{\beta}\psi$, the set of indices β must be contained in the set of indices α, and $m_{\mathbb{J}}^{\mathbb{K}}(\phi) \leq m_{\mathbb{J}}^{\mathbb{K}}(\psi)$. Since ψ'' occurs in $\mathrm{ind}_H^G(\phi'')$, since ${}^{\alpha}\phi$ is the unique Galois conjugate of ϕ overlapping with ${}^{\alpha}\psi$, and since ${}^{\alpha}\phi$ induces to ${}^{\alpha}\psi$, the set of indices α must be contained in the set of indices β, and $m_{\mathbb{J}}^{\mathbb{K}}(\phi) \geq m_{\mathbb{J}}^{\mathbb{K}}(\psi)$. So the two sets of indices coincide. That is to say, $\mathbb{Q}[\psi''] = \mathbb{Q}[\phi'']$. Furthermore, $m_{\mathbb{J}}^{\mathbb{K}}(\phi) = m_{\mathbb{J}}^{\mathbb{K}}(\psi)$. It follows that ϕ'' induces to ψ''. Also, ϕ'' occurs only once in the restriction of ψ, in other words, the induction from ϕ'' to ψ'' is shallow. We have already observed that $\mathbb{Q}[\psi''] = \mathbb{Q}[\phi'']$, in other words, the induction is narrow. Thus, (b) implies (a).

Still assuming (b), we now want (c). Each $\mathbb{L}H$-irrep contained in ϕ must overlap with at least one $\mathbb{L}G$-irrep contained in ψ, and each $\mathbb{L}G$-irrep contained in ψ must overlap with at least one $\mathbb{L}H$-irrep contained in ϕ. Applying the functor $\mathbb{L} \otimes_{\mathbb{K}} —$ to the $\mathbb{K}$-algebra isomorphism $\mathrm{End}_{\mathbb{K}H}(\phi) \cong \mathrm{End}_{\mathbb{K}G}(\psi)$, we obtain an isomorphism of semisimple rings $\mathrm{End}_{\mathbb{L}H}(\mathbb{L}\phi) \cong \mathrm{End}_{\mathbb{L}G}(\mathbb{L}\psi)$. The number of Wedderburn components of this semisimple ring is equal to the number of distinct $\mathbb{L}H$-irreps contained in ϕ, and it is also equal to the number of distinct $\mathbb{L}G$-irreps contained in ψ. By Theorem 2.2, all the $\mathbb{L}H$-irreps contained in ϕ have the same multiplicity m, and all the $\mathbb{L}G$-irreps contained in ψ have the same multiplicity n. So the Wedderburn components all have the same degree as matrix algebras over their associated division rings, and $m = n$. It follows that there is a bijection $\psi' \leftrightarrow \phi'$ whereby ψ' is shallowly induced from ϕ'. We must show that the induction is narrow. By the formula for induction of characters, the field $\mathbb{Q}[\phi']$ (which is independent of the choice of ϕ') contains the field $\mathbb{Q}[\psi']$ (which is independent of the choice of ψ'). By Theorem 2.2, the number of distinct $\mathbb{L}H$-irreps contained in ϕ is equal to the order of the Galois group $\mathrm{Gal}(\mathbb{Q}[\phi']/\mathbb{Q}[\phi])$ while the number of distinct $\mathbb{L}G$-irreps contained in ψ is the order of $\mathrm{Gal}(\mathbb{Q}[\psi']/\mathbb{Q}[\psi])$. But we already know that these two numbers are equal. Moreover, $\mathbb{Q}[\psi] = \mathbb{Q}[\phi]$ as part of the hypothesis on ψ and ϕ. Therefore $\mathbb{Q}[\psi'] = \mathbb{Q}[\phi']$. We have gotten (c) from (b). To obtain (b) from (a) or from (c), we interchange the extensions $\mathbb{L}/\mathbb{K}$ and $\mathbb{K}/\mathbb{J}$. □

For facility of use, it is worth restating the theorem.

Theorem 3.5. (Field-Changing Theorem) *Let $H \leq G$ and let ψ be a $\mathbb{K}G$-irrep induced from a $\mathbb{K}H$-irrep ϕ. Let $\mathbb{L}$ be any field having characteristic zero. Then the following conditions are equivalent:*

(a) *ψ is tightly induced from ϕ.*

(b) *$\psi_{\mathbb{Q}}$ is tightly induced from $\phi_{\mathbb{Q}}$.*

(c) *The $\mathbb{L}G$-irreps ψ' that are quasiconjugate to ψ are in a bijective correspondence with the $\mathbb{L}H$-irreps ϕ' that are quasiconjugate to ϕ. They correspond $\psi' \leftrightarrow \phi'$ when ψ' is tightly induced from ϕ'.*

The theorem tells us that, in some sense, the genetic theory is independent of the coefficient field $\mathbb{K}$. Condition (b) is a useful theoretical criterion for tightness of a given induction $\psi = \mathrm{ind}_H^G(\phi)$. It sometimes allows us to generalize immediately from the case $\mathbb{K} = \mathbb{Q}$ to the case where $\mathbb{K}$ is arbitrary; see the next section. However, the rational irreps of a given finite p-group are usually very difficult to determine. For explicit analysis of concrete examples, a more practical criterion for tightness is given by the following corollary.

Corollary 3.6. *Let $H \leq G$ and let ψ be a $\mathbb{K}G$-irrep induced from a $\mathbb{K}H$-irrep ϕ. Then the vertex fields satisfy the inequality $\mathbb{V}(\psi) \leq \mathbb{V}(\phi)$, and equality holds if and only if the induction is tight.*

Proof. By passing from $\mathbb{K}$ to the algebraic closure of $\mathbb{K}$, thence to the splitting field $\mathbb{Q}_{n(G)}$, thence to $\mathbb{C}$, we see that $\psi' = \mathrm{ind}_H^G(\phi')$ for some complex irreps ψ' and ϕ' quasiconjugate to ψ and ϕ. By the formula for induction of characters, the vertex field $\mathbb{V}(\psi) = \mathbb{Q}[\psi']$ is contained in the vertex field $\mathbb{V}(\phi) = \mathbb{Q}[\phi']$. By the Field-Changing Theorem, ψ is tightly induced from ϕ if and only if ψ' is tightly induced from ϕ'. For complex irreps, tight induction is just narrow induction. □

Let us give an example. For $n \geq 5$, we define $\mathrm{DD}_{2^n} = V_4 \rtimes C_{2^{n-2}}$ as a semidirect product where V_4 acts faithfully. The 2-group DD_{2^n} has generators a, b, c, d with relations

$$a^{4u} = b^2 = c^2 = d^2 = bcd = 1\,, \qquad bab^{-1} = a^{-1}\,, \qquad cac^{-1} = a^{2u+1}\,, \qquad dad^{-1} = a^{2u-1}$$

where $u = 2^{n-4}$. Fixing n, let us write $\mathrm{DD} = \mathrm{DD}_{2^n}$. Let ω be a primitive $4u$-th root of unity. The subgroup $A = \langle a \rangle \cong C_{4u}$ has complex irrep η such that $\eta(a) = \omega$. The subgroup $D' = \langle b, a \rangle \cong D_{8u}$ has a real abirrep (absolutely irreducible representation) ϕ' such that $\mathbb{C}\phi' = \mathrm{ind}_A^{D'}(\eta)$. Using Lemma 2.5, we see that DD has a real abirrep $\chi = \mathrm{ind}_{D'}^{DD}(\phi')$, and furthermore, the faithful real irreps of DD are precisely the Galois conjugates of χ. The induction from ϕ' to χ is not tight. One way to see this is to calculate the vertex fields of ϕ' and χ over $\mathbb{Q}$. The character values vanish of A and, given an integer k, we have $\phi'(a^k) = \omega^k + \omega^{-k}$ and $\chi(a^k) = \omega^k + \omega^{k(2u-1)} + \omega^{k(2u+1)} + \omega^{-k}$. But $\omega^{2u} = -1$ so χ vanishes off $\langle a^2 \rangle$ and $\chi(a^{2k}) = 2(\omega^{2k} + \omega^{-2k})$. Since ϕ' and χ are absolutely irreducible, the vertex fields are $\mathbb{V}(\phi') = \mathbb{Q}[\phi'] = \mathbb{Q}[\omega + \omega^{-1}]$ and $\mathbb{V}(\chi) = \mathbb{Q}[\chi] = \mathbb{Q}[\omega^2 + \omega^{-2}]$; the former is a quadratic extension of the latter. Alternatively, to see directly that the induction is shallow but not narrow, observe that $\mathrm{res}_{D'}^{DD}(\chi) = \phi' + {}^{\alpha}\phi'$ where α is any Galois automorphism sending ω to $-\omega$ or to $-\omega^{-1}$. However, χ is tightly induced from a strict subgroup. Consider the subgroups

$$C = \langle c \rangle \cong C_2\,, \qquad D = \langle a^2, b \rangle \cong D_{4u}\,, \qquad T = \langle a^2, b, c \rangle = C \times D\,.$$

Let ϕ be the real abirrep of T such that $\phi(a^2) = \omega^2 + \omega^{-2}$ and $\phi(b) = 0$ and $\phi(c) = 2$. Thus, $\mathrm{Ker}(\phi) = C$ and ϕ is the inflation of a faithful real abirrep of the group $T/C \cong D_{4u}$. Direct calculation yields $\chi = \mathrm{ind}_T^{DD}(\phi)$. This induction is tight because, by the absolute irreducibility of ϕ, the vertex field is $\mathbb{V}(\phi) = \mathbb{Q}[\phi] = \mathbb{Q}[\omega^2 + \omega^{-2}] = \mathbb{V}(\chi)$. We shall be returning to this example at the end of the next section.

4 Genotypes and germs

Let us begin by quickly proving the Genotype Theorem 1.1. The Field-Changing Theorem 3.5 implies that, for any subgroup $H \leq G$, a given $\mathbb{K}G$-irrep ψ is tightly induced from H if and only if the $\mathbb{Q}G$-irrep $\psi_{\mathbb{Q}}$ is tightly induced from H. Moreover, for any $\mathbb{K}H$-irrep ϕ that tightly induces to ψ, the $\mathbb{Q}H$-irrep $\phi_{\mathbb{Q}}$ tightly induces to $\phi_{\mathbb{Q}}$. Letting K be the kernel of ϕ, then K is the kernel of any Galois conjugate of ϕ and, via Theorem 2.2, K is the kernel of $\phi_{\mathbb{Q}}$. We deduce that the subquotients from which ψ is tightly induced coincide with the subquotients from which $\psi_{\mathbb{Q}}$ is tightly induced. The Genotype Theorem thus reduces to the case $\mathbb{K} = \mathbb{Q}$. In that special case, the theorem was obtained by Bouc [2, 3.4, 3.6, 3.9, 5.9]. Alternatively, a similar use of the Field Changing Theorem reduces to the case $\mathbb{K} = \mathbb{C}$, and in that special case, the theorem was obtained by Kronstein [12, 2.5]. The proof of the Genotype Theorem is complete.

The existence half of the Genotype Theorem is equivalent to the following result, which is due to Roquette [14] in the case $\mathbb{K} = \mathbb{Q}$ and to Kronstein [12] in the case $\mathbb{K} = \mathbb{C}$.

Theorem 4.1. *Given a $\mathbb{K}G$-irrep ψ, then is not tightly induced from any strict subquotient of G if and only if G is Roquette and ψ is faithful.*

We shall give a direct proof of Theorem 4.1 without invoking the Field-Changing Theorem. The direct proof will yield a recursive algorithm for finding the genotype and a germ for a given irrep. The argument is adapted from Hambleton–Taylor–Williams [10] and Bouc [2]. Before presenting two preparatory lemmas, let us make a preliminary claim: supposing that G is non-cyclic and abelian, then G has no faithful $\mathbb{K}$-irreps. To demonstrate the claim, consider a $\mathbb{K}G$-irrep ψ. Letting $\mathbb{L}$ be a splitting field for $\mathbb{K}$, then $\mathbb{L}\psi$ is a direct sum of mutually Galois conjugate $\mathbb{L}G$-irreps. All of those $\mathbb{L}G$-irreps have the same kernel K. The hypothesis that G is abelian implies that the $\mathbb{L}G$-irreps in question are 1-dimensional, hence G/K is cyclic. The hypothesis that G is non-cyclic implies that $K \neq 1$. But K must also be the kernel of ψ. Therefore ψ is non-faithful.

Lemma 4.2. *Suppose that G is non-Roquette and that there exists a faithful $\mathbb{K}G$-irrep ψ. Then there exists a normal subgroup E of G such that $E \cong C_p \times C_p$ and $E \cap Z(G) \cong C_p$. For any such E, the subgroup $T = C_G(E)$ is maximal in G. Letting ϕ be any $\mathbb{K}T$-irrep overlapping with ψ, then ψ is tightly induced from ϕ.*

Proof. The argument is essentially in [10, 2.16] and [2, 3.4], but we must reproduce the constructions in order to check the tightness of the induction. First observe that, given any normal non-cyclic abelian subgroup A of G, then the restriction of ψ to A is faithful, whence the preliminary claim tells us that any $\mathbb{K}A$-irrep overlapping with ψ must be non-inertial. The center $Z(G)$ is cyclic because every $\mathbb{K}Z(G)$-irrep is inertial in G. Let Z be the subgroup of $Z(G)$ with order p. let B be the maximal elementary abelian subgroup of A. Then $Z \lhd B \trianglelefteq G$ and B/Z intersects non-trivially with the centre of G/Z, so there exists an intermediate subgroup $Z \leq E \leq B$ such that E/Z is a central subgroup of G/Z with order p. Plainly, E satisfies the required conditions. The non-trivial p-group G/T embeds in the group $\mathrm{Aut}(C_p \times C_p) = \mathrm{GL}_2(p)$, which has order $p(p-1)(p^2-1)$. So T is maximal in G.

Let ϵ_1 be a $\mathbb{K}E$-irrep overlapping with ψ. The preliminary claim implies that the inertia group of ϵ_1 is a strict subgroup of G. On the other hand, the inertia group must contain the centralizer T of E. But T is maximal. So T is the inertia group of ϵ_1. The $\mathbb{K}E$-irreps overlapping with ψ are precisely the G-conjugates of ϵ, and we can number them as ϵ_1, ...,

ϵ_p because $p = |G : T|$. The proof of the preliminary claim reveals that the kernels of ϵ_1, ..., ϵ_p are mutually distinct; the kernels are non-trivial yet their intersection is trivial. In particular, ϵ_1, ..., ϵ_p belong to mutually distinct Galois conjugacy classes. By Clifford theory, $\text{res}^G_T(\psi) = \phi_1 + ... + \phi_p$ as a direct sum of $\mathbb{K}T$-irreps such that each ϕ_j restricts to a multiple of ϵ_j. Therefore, ϕ_1, ..., ϕ_p are mutually distinct and, in fact, they belong to mutually distinct Galois conjugacy classes. It follows that each ϕ_j induces tightly to ψ. □

Lemma 4.3. *Let A be a self-centralizing normal cyclic subgroup of G and let $A \leq H < G$. Then no faithful $\mathbb{K}G$-irrep is tightly induced from H.*

Proof. Deny, and consider a faithful $\mathbb{K}G$-irrep ψ that is tightly induced from a $\mathbb{K}H$-irrep ϕ of H. By Remark 3.3, we may assume that H is maximal in G. In particular, $H \trianglelefteq G$. So $\text{res}^G_H(\psi) = \phi_1 + ... + \phi_p$ as a sum of G-conjugates of ϕ. Since A is self-centralizing in both G and H, Lemma 2.5 implies that ϕ_1, ..., ϕ_p are faithful. Lemma 2.6 implies that ϕ_1, ..., ϕ_p are Galois conjugates. This contradicts the tightness of the induction from ϕ. □

In one direction, Theorem 4.1 is immediate from Lemma 4.2. To complete the direct proof of the theorem, it remains only to show that, supposing G is Roquette and letting ψ be a faithful $\mathbb{K}G$-irrep, then ψ is not tightly induced from a strict subgroup. Our argument is close to [10, 2.15], but with some modification (their appeal to the uniqueness of the "basic representation" does not generalize). For a contradiction, suppose that ψ is tightly induced from a $\mathbb{K}H$-irrep ϕ where $H < G$. Again, by Remark 3.3, we may assume that $|G : H| = p$. By Roquette's Classification Theorem 2.4, G has a self-centralizing cyclic subgroup A with index 1 or p. Plainly, G cannot be cyclic. So $|G : A| = p$. By Lemma 4.3, $H \neq A$. So the subgroup $B = A \cap H$ has index p^2 in G.

First suppose that B is not self-centralizing in H. Then H must be abelian. But G is Roquette, hence H is cyclic. Also, G is non-abelian, so H is self-centralizing. This contradicts Lemma 4.3. Now suppose that B is self-centralizing in H. By Lemma 2.5, there exists a faithful $\mathbb{K}B$-irrep ξ such that $\text{ind}^H_B(\xi)$ is a multiple of ϕ. Letting $\zeta = \text{ind}^A_B(\xi)$, then $\text{ind}^G_A(\zeta)$ is a multiple of ψ. Every $\mathbb{K}A$-irrep occurring in ζ must also occur in $\text{res}^G_A(\psi)$. But ζ is induced from B, so some non-faithful $\mathbb{K}A$-irrep η must occur in ζ. Perforce, η occurs in $\text{res}^G_A(\psi)$. This contradicts part (1) of Lemma 2.5. The direct proof of Theorem 4.1 is finished.

Lemma 4.2 gives an algorithm for finding a genetic subquotient and a germ. First we replace G with $G/\text{Ker}(\psi)$ to reduce to the case where ψ is faithful. If $G/\text{Ker}(\psi)$ is Roquette, then $G/\text{Ker}(\psi)$ is a genetic subquotient, ψ is a germ, and the algorithm terminates. Otherwise, in the notation of the lemma, we replace G and ψ with T and ϕ, respectively, and we repeat the process.

By the way, the noncyclic abelian subgroup E is central in T, so the $\mathbb{K}T$-irrep ϕ is never faithful, and we deduce the second part of the following incidental corollary. The first part of the corollary is immediate from the Genotype Theorem 1.1.

Corollary 4.4. *Let H/K and H'/K' be genetic factors for the same $\mathbb{K}G$-irrep. Then $|H| = |H'|$ and $|K| = |K'|$. Furthermore, $|G : H| \leq |K|$.*

Let us end this section with a reassessment of the example $\text{DD} = \text{DD}_{2^n} = \text{DD}_{16u}$, which was discussed at the end of the previous section. We employ the same notation as before. Recall that, although the $\mathbb{R}\,\text{DD}$-irrep χ is induced from the subgroup $D' \cong D_{4u}$, the induction is not tight. The failure of tightness can now be seen straight from Lemma 4.3 because D' contains the self-centralizing normal cyclic subgroup A. We have already seen that $\chi = \text{ind}^G_T(\phi)$ and that ϕ is inflated from a faithful $\mathbb{R}$-irrep of the subquotient $T/C \cong D_{4u}$. So, if $n \geq 6$, then T/C is a genetic subquotient and ϕ is a germ. In particular, the genetic type is $\mathrm{Typ}(\chi) = D_{2^{n-2}}$, except in the case $n = 5$, and in that case, $\mathrm{Typ}(\chi) = C_2$. But let us recover these conclusions from the algorithm in a methodical way. As we noted in Section 2, the 2-group $\mathrm{Mod}_{8u} = \langle a, c \rangle$ has a characteristic subgroup $E = \langle a^{2u}, c \rangle \cong V_4$. Treating Mod_{8u} as a maximal subgroup of DD, then E is normal in DD, and the subgroup $T = C_G(E)$ and the irrep ϕ that appear in Lemma 4.2 coincide with the subgroup $T = \langle a^2, b, c \rangle$ and the irrep ϕ which we considered at the end of Section 3. Noting that $C = \mathrm{Ker}(\phi)$, we again arrive at the conclusion that, if $n \geq 6$ then T/C is a genetic subquotient and ϕ is a germ. Of course, when $n = 5$, the algorithm continues, the second iteration replacing the faithful $\mathbb{R}D_8$-irrep with the faithful $\mathbb{R}C_2$-irrep. Let us point out that, in Section 3, we calculated the vertex fields χ and ϕ in order to show that the induction $\chi = \mathrm{ind}_{T/C}^{DD}(\phi)$ is tight. We have now dispensed with that trip, and the tightness has been delivered to us as part of the conclusion of Lemma 4.2.

5 Characterizations of the genotype

In the first movement, we shall confine our attention to the Roquette p-groups. For those p-groups, we shall calculate some of the invariants that were listed in Section 2. In the second movement, we shall show that all the invariants listed in Section 2 are genetic invariants. We shall also terminate a couple of loose-ends concerning well-definedness. The third movement will address two questions that were raised in Section 1: How can the genotype $\mathrm{Typ}(\psi)$ be ascertained from easily calculated genetic invariants such as the order $v(\psi)$, the vertex set $\mathrm{Vtx}(\psi)$, the vertex field $\mathbb{V}(\psi)$, the Frobenius–Schur type $\Delta(\psi)$? How can $\mathrm{Typ}(\psi)$ be used to ascertain less tractable genetic invariants such as the exponent $n(\psi)$, the minimal splitting fields, the Fein field?

To open the first movement, let us observe that, when p is odd, there is nothing much to say, as in the next lemma. Note that, given a $\mathbb{K}G$-irrep ψ, then $\mathbb{V}(\psi)$ is a splitting field for $\psi_{\mathbb{Q}}$ if and only if $m(\psi) = 1$. For the time-being, we shall understand a Fein field for ψ to be a field that is both a subfield of $\mathbb{Q}_{n(\psi)}$ and also a splitting field for $\psi_{\mathbb{Q}}$. When we have established the existence and uniqueness of the Fein field in Theorem 5.7, we shall be at liberty to write the Fein field as $\mathrm{Fein}(\psi)$.

Lemma 5.1. *Suppose that p is odd. Let m be a positive integer. Let ψ be a faithful $\mathbb{K}$-irrep of the cyclic group C_{p^m}. Then the order of ψ is $v(\psi) = p^m - p^{m-1}$. The exponent of ψ is $n(\psi) = p^m$. The unique minimal splitting field for $\psi_{\mathbb{Q}}$ is the unique Fein field for ψ, and it coincides with the vertex field $\mathbb{V}(\psi) = \mathbb{Q}_{p^m}$. The Schur index is $m(\psi) = 1$. The Frobenius–Schur type is $\Delta(\psi) = \mathbb{C}$. The vertex set $\mathrm{Vtx}(\psi)$ is free and transitive as permutation set for the Galois group $\mathrm{Aut}(\mathbb{Q}_{p^m}) = \mathrm{Gal}(\mathbb{Q}_{p^m}) \cong \mathrm{Aut}(C_{p^m}) \cong (\mathbb{Z}/p^m)^{\times}$.*

We refrain from a systematic discussion of the groups C_1 and C_2. The only Roquette p-groups left are the Roquette 2-groups with more than one faithful complex irrep. These will be covered by the next four lemmas. The lemmas are inevitable exercises, hence they are well-known. If one could collate a trawl of citations encompassing all the conclusions, then that would be a gnomic achievement. We mention that some of the material — including a rather different discussion of minimal splitting fields for the quaternion groups — can be found in Leedham-Green–McKay [13, 10.1.17].

Let us throw some more notation. We shall be making use of the matrices

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

We define $\mathrm{ex}(t) = e^{2\pi i t}$ and $\mathrm{cs}(t) = \cos(2\pi t)$ and $\mathrm{sn}(t) = \sin(2\pi t)$ for $t \in \mathbb{R}$. The matrices

$$R(t) = \begin{pmatrix} \mathrm{cs}(t) & -\mathrm{sn}(t) \\ \mathrm{sn}(t) & \mathrm{cs}(t) \end{pmatrix} , \quad I(t) = i \begin{pmatrix} \mathrm{sn}(t) & \mathrm{cs}(t) \\ -\mathrm{cs}(t) & \mathrm{sn}(t) \end{pmatrix} , \quad S(t) = \begin{pmatrix} \mathrm{cs}(t) & i\,\mathrm{sn}(t) \\ i\,\mathrm{sn}(t) & \mathrm{cs}(t) \end{pmatrix}$$

satisfy the relation $R(t+t') = R(t)R(t')$ and similarly for $I(t+t')$ and $S(t+t')$. Let

$$A_s(t) = \begin{pmatrix} \mathrm{cs}(t) + i\,\mathrm{sn}(t)/\mathrm{cs}(s) & \mathrm{sn}(t)\mathrm{sn}(s)/\mathrm{cs}(s) \\ \mathrm{sn}(t)\mathrm{sn}(s)/\mathrm{cs}(s) & \mathrm{cs}(t) - i\,\mathrm{sn}(t)/\mathrm{cs}(s) \end{pmatrix}$$

where $s \in \mathbb{R}$. By direct calculation, $A_s(t+t') = A_s(t)A_s(t')$.

Let v be a power of 2 with $v \geq 2$. For convenience, we embed $\mathbb{Q}_{4v}$ in $\mathbb{C}$ by making the identification $\mathbb{Q}_{4v} = \mathbb{Q}[\omega]$ where $\omega = \mathrm{ex}(1/4v)$. The Galois group

$$\mathrm{Aut}(\mathbb{Q}_{4v}) = \mathrm{Gal}(\mathbb{Q}_{4v} : \mathbb{Q}) \cong \mathrm{Aut}(C_{4v}) \cong (\mathbb{Z}/4v)^{\times} \cong C_2 \times C_v$$

has precisely 3 involutions, namely β, γ, δ which act on $\mathbb{Q}_{4v}$ by

$$\beta(\omega) = \omega^{-1} , \qquad \gamma(\omega) = \omega^{2v+1} = -\omega , \qquad \delta(\omega) = \omega^{2v-1} = -\omega^{-1} .$$

For a subgroup $\mathcal{H} \leq \mathrm{Aut}(\mathbb{Q}_{4v})$, we let $\mathrm{Fix}(\mathcal{H})$ be the intermediate subfield $\mathbb{Q} \leq \mathrm{Fix}(\mathcal{H}) \leq \mathbb{Q}_{4v}$ fixed by $\mathcal{H}$. A straightforward application of the Fundamental Theorem of Galois Theory shows that $\mathbb{Q}_{4v}$ has precisely 3 maximal subfields, namely

$$\begin{aligned} \mathbb{Q}_{4v}^{\mathbb{R}} &= \mathrm{Fix}\langle\beta\rangle = \mathbb{Q}[\omega + \omega^{-1}] = \mathbb{Q}[\mathrm{cs}(r/4v)] = \mathbb{Q}[\mathrm{sn}(r/4v)] = \mathbb{R} \cap \mathbb{Q}_{4v} , \\ \mathbb{Q}_{2v} &= \mathrm{Fix}\langle\gamma\rangle = \mathbb{Q}[\omega^2] , \\ \mathbb{Q}_{4v}^{\mathbb{I}} &= \mathrm{Fix}\langle\delta\rangle = \mathbb{Q}[\omega - \omega^{-1}] = \mathbb{Q}[i\,\mathrm{cs}(r/4v)] = \mathbb{Q}[i\,\mathrm{sn}(r/4v)] . \end{aligned}$$

Here, r is any odd integer. These three subfields all have index 2 in $\mathbb{Q}_{4v}$. In other words, they have degree v over $\mathbb{Q}$. Glancing back at the proof of Theorem 2.4, we observe that β and δ have no square root in $\mathrm{Aut}(\mathbb{Q}_{4v})$. So γ belongs to every non-trivial subgroup of $\mathrm{Aut}(\mathbb{Q}_{4v})$ except for $\langle\beta\rangle$ and $\langle\delta\rangle$. Therefore $\mathbb{Q}_{2v}$ contains every strict subfield of $\mathbb{Q}_{4v}$ except for $\mathbb{Q}_{4v}^{\mathbb{R}}$ and $\mathbb{Q}_{4v}^{\mathbb{I}}$. These observations yield a complete description of the intermediate subfields $\mathbb{Q} \leq K \leq \mathbb{Q}_{4v}$. (In particular, we see that, letting u be any power of 2 with $2 \leq u \leq v$, then there are precisely three intermediate fields with degree u over $\mathbb{Q}$. But there are four families of Roquette 2-groups: cyclic, dihedral, semidihedral, quaternion. This already suggests that distinguishing between the four families may be little awkward.)

In the following four lemmas, we still let v be a power of 2 with $v \geq 2$. The first one is similar to Lemma 5.1, and again, it is obvious. We postpone discussion of the vertex set.

Lemma 5.2. *Let ψ be a faithful $\mathbb{K}$-irrep of the cyclic group C_{2v}. Then the order is $v(\psi) = v$. The exponent is $n(\psi) = 2v$. The unique minimal splitting field for $\psi_{\mathbb{Q}}$ is the unique Fein field for ψ, and it coincides with the vertex field $\mathbb{V}(\psi) = \mathbb{Q}_{2v}$. The Schur index is $m(\psi) = 1$. The Frobenius–Schur type is $\Delta(\psi) = \mathbb{C}$.*

Lemma 5.3. *Let ψ be a faithful $\mathbb{K}$-irrep of the dihedral group D_{8v}. Then $v(\psi) = v$ and $n(\psi) = 4v$. The unique minimal splitting field for $\psi_{\mathbb{Q}}$ is the unique Fein field for ψ, and it coincides with the vertex field $\mathbb{V}(\psi) = \mathbb{Q}^{\mathbb{R}}_{4v}$. The Schur index is $m(\psi) = 1$. The Frobenius–Schur type is $\Delta(\psi) = \mathbb{R}$.*

Proof. Plainly, $v(\psi) = v$. Employing the standard presentation, the group $D_{8v} = \langle a, b \rangle$ has a faithful irreducible matrix representation ψ given by $a \mapsto R(1/4q)$ and $b \mapsto B$. By considering the matrix entries, we see that ψ is affordable over the field $\mathbb{Q}^{\mathbb{R}}_{4v}$. Hence $\mathbb{V}(\psi) \leq \mathbb{Q}^{\mathbb{R}}_{4v}$. But we must have equality, because the character value at a is $\psi(a) = 2\text{cs}(1/4v)$, which is a primitive element of $\mathbb{Q}^{\mathbb{R}}_{4v}$. It is clear that ψ has all the specified properties. All of these properties are invariant under Galois conjugation. So, invoking Corollary 2.8, the properties hold for any faithful $\mathbb{K}D_{8v}$-irrep. □

Lemma 5.4. *Let ψ be a faithful $\mathbb{K}$-irrep of the semidihedral group SD_{8v}. Then $v(\psi) = v$ and $n(\psi) = 4v$. The unique minimal splitting field for $\psi_{\mathbb{Q}}$ is the unique Fein field for ψ, and it coincides with the vertex field $\mathbb{V}(\psi) = \mathbb{Q}^{\mathbb{I}}_{4v}$. The Schur index is $m(\psi) = 1$. The Frobenius–Schur type is $\Delta(\psi) = \mathbb{C}$.*

Proof. The argument is similar to the proof of the previous lemma. Note that SD_{8v} has a faithful irreducible matrix representation the standard generators a and d to the matrices $I(1/4q)$ and D, respectively. □

Lemma 5.5. *Let ψ be a faithful $\mathbb{K}$-irrep of the quaternion group Q_{8v}. Then $v(\psi) = v$ and $n(\psi) = 4v$. The vertex field is $\mathbb{V}(\psi) = \mathbb{Q}^{\mathbb{R}}_{4v}$. The unique Fein field for ψ is $\mathbb{Q}_{4v}$. Two non-isomorphic minimal splitting fields for $\psi_{\mathbb{Q}}$ are $\mathbb{Q}_{4v}$ and $\mathbb{Q}^{\mathbb{I}}_{8v}$. The Schur index is $m(\psi) = 2$. The Frobenius–Schur type is $\Delta(\psi) = \mathbb{H}$.*

Proof. By Corollary 2.8 again, we may assume that ψ is the faithful $\mathbb{C}Q_{4v}$-irrep such that $\psi(a^r) = \omega^r + \omega^{-r} = 2\text{cs}(r/4v)$ for $r \in \mathbb{Z}$. There is a matrix representation of ψ such that the standard generators a and x act as $S(1/4q)$ and X, respectively. From the character values, we see that $\mathbb{V}(\psi) = \mathbb{Q}^{\mathbb{R}}_{4v}$. Since the dihedral groups are the only non-abelian finite groups with a faithful representation on the Euclidian plane, ψ is not affordable over $\mathbb{R}$. (Alternatively, we can observe that $\sum_{f \in A} \psi(f^2) = 0$ and $\psi(g^2) = \psi(a^{2q}) = -2$ for $g \in Q_{4v} - \langle a \rangle$, whence $\text{fs}(\psi) = -1$.) Perforce, ψ is not affordable over $\mathbb{Q}^{\mathbb{R}}_{4v}$. On the other hand, by considering the matrix entries of $S(1/4q)$, we see that ψ is affordable over $\mathbb{Q}_{4v}$. (Alternatively, we can appeal to Brauer's Splitting Theorem.) The quadratic extension $\mathbb{Q}_{4v}$ of $\mathbb{Q}^{\mathbb{R}}_{4v}$ must be a minimal splitting field for ψ. It follows that $n(\psi) = 4v$ and $\mathbb{Q}_{4v}$ is the unique Fein field of ψ. These observations also imply that $m(\psi) = |\mathbb{Q}_{4v} : \mathbb{Q}^{\mathbb{R}}_{4v}| = 2$ and $\Delta(\psi) = \mathbb{H}$.

By direct calculation, it is easy to check that ψ has another matrix representation given by $a \mapsto A_{1/8v}(1/4v)$ and $x \mapsto X$. The field generated by the matrix entries of $A_{1/8v}(1/4v)$ is $\mathbb{Q}[\text{cs}(1/4v), \text{sn}(1/4v), i\,\text{cs}(1/8v), i\,\text{sn}(1/8v)] = \mathbb{Q}^{\mathbb{I}}_{8v}$, and this must be a minimal splitting field because it is a quadratic extension of the vertex field. The two minimal splitting fields that we have mentioned are non-isomorphic because they are distinct subfields of the cyclotomic field $\mathbb{Q}_{8v}$, whose Galois group over $\mathbb{Q}$ is abelian. □

We now discuss the vertex sets for the faithful irreps of the Roquette 2-groups. We continue to assume that v is a power of 2 with $v \geq 2$. Below, we shall find that, if $p = 2$ and if ψ is a $\mathbb{K}G$-irrep with order $v(\psi) = v$, then there are precisely four possibilities for the genotype $\text{Typ}(\psi)$, namely C_{2v}, D_{8v}, SD_{8v}, Q_{8v}. To what extent can we distinguish between these four possibilities by considering Galois actions on the vertices? Recall, from Section 2, that the vertex set $\mathrm{Vtx}(\psi)$ is a permutation set for the Galois group of a sufficiently large Galois extension of $\mathbb{Q}$. The question will reduce to a consideration of the Roquette 2-groups. Let ψ_C, ψ_D, ψ_S, ψ_Q be faithful $\mathbb{K}$-irreps of C_{2v}, D_{8v}, SD_{8v}, Q_{8v}, respectively. Since $n(\psi) = 2v$, we can regard $\mathrm{Vtx}(\psi_C)$ as a permutation set for the Galois group $\mathrm{Aut}(\mathbb{Q}_{2v}) = \mathrm{Gal}(\mathbb{Q}_{2v}/\mathbb{Q})$. More generally, we can regard $\mathrm{Vtx}(\psi)$ as a permutation set for $\mathrm{Aut}(\mathbb{Q}_n)$ where n is any multiple of $2v$. Meanwhile, since $n(\psi_D) = n(\psi_S) = n(\psi_Q) = 4v$, we can regard $\mathrm{Vtx}(\psi_D)$ and $\mathrm{Vtx}(\psi_S)$ and $\mathrm{Vtx}(\psi_Q)$ as permutation sets for $\mathrm{Aut}(\mathbb{Q}_{4v})$ and, more generally, as permutation sets for $\mathrm{Aut}(\mathbb{Q}_n)$ where n is now any multiple of $4v$. In view of these observations, we put $n = 4v$. We regard all four vertex sets $\mathrm{Vtx}(\psi_C)$, $\mathrm{Vtx}(\psi_D)$, $\mathrm{Vtx}(\psi_S)$, $\mathrm{Vtx}(\psi_Q)$ as $\mathrm{Aut}(\mathbb{Q}_{4v})$-sets. As we noted in Section 2, all four of them are transitive. Since $\mathrm{Aut}(\mathbb{Q}_{4v})$ has size $2v$ and since the four vertex sets all have size v, the four point-stabilizer subgroups all have size 2. Of course, since $\mathrm{Aut}(\mathbb{Q}_{4v})$ is abelian, any transitive $\mathrm{Aut}(\mathbb{Q}_{4v})$-set has a unique point-stabilizer subgroup.

Lemma 5.6. *With the notation above, the vertex sets* $\mathrm{Vtx}(\psi_C)$ *and* $\mathrm{Vtx}(\psi_D)$ *and* $\mathrm{Vtx}(\psi_S)$ *and* $\mathrm{Vtx}(\psi_Q)$ *are transitive* $\mathrm{Aut}(\mathbb{Q}_{4v})$*-sets, and the point-stabilizer subgroups are* $\langle\gamma\rangle$ *and* $\langle\beta\rangle$ *and* $\langle\delta\rangle$ *and* $\langle\beta\rangle$*, respectively.*

Proof. We apply the Fundamental Theorem of Galois Theory to the Galois group $\mathrm{Aut}(\mathbb{Q}_{4v})$ of the field extension $\mathbb{Q}_{4v}/\mathbb{Q}$. The subgroups $\langle\gamma\rangle$ and $\langle\beta\rangle$ and $\langle\delta\rangle$ and $\langle\beta\rangle$ are the centralizers of the subfields $\mathbb{V}(\psi_C) = \mathbb{Q}_{2v}$ and $\mathbb{V}(\psi_D) = \mathbb{Q}_{4v}^{\mathbb{R}}$ and $\mathbb{V}(\psi_S) = \mathbb{Q}_{4v}^{\mathbb{I}}$ and $\mathbb{V}(\psi_Q) = \mathbb{Q}_{4v}^{\mathbb{R}}$, respectively. □

To begin the slow movement, let us recall some obligations from Section 2. There, we listed some invariants of a $\mathbb{K}G$-irrep ψ, and we stated that they are genetic invariants. We also stated that there exists a unique Fein field for ψ. We indicated that we would recover Schilling's Theorem. We stated that the vertex set $\mathrm{Vtx}(\psi)$ is the maximum field that embeds in every splitting field for $\psi_{\mathbb{Q}}$. In the next few results, we shall prove those assertions.

Theorem 5.7. *Let* ψ *be a* $\mathbb{K}G$*-irrep. Let* $\mathbb{L}/\mathbb{J}$ *be a characteristic zero field extension. Then the* $\mathbb{L}/\mathbb{J}$*-relative Schur index* $m_{\mathbb{J}}^{\mathbb{L}}(\psi)$*, the* $\mathbb{L}/\mathbb{J}$*-relative order* $v_{\mathbb{J}}^{\mathbb{L}}(\psi)$*, the* $\mathbb{J}$*-algebra isomorphism class of the endomorphism ring* $\mathrm{End}_{\mathbb{J}G}(\psi)$*, the class of minimal splitting fields for* $\psi_{\mathbb{J}}$ *and the* $\mathbb{J}$*-relative vertex field* $\mathbb{V}_{\mathbb{J}}(\psi)$ *are genetic invariants of* ψ*. In particular,* $m_{\mathbb{J}}(\psi)$*,* $v_{\mathbb{J}}(\psi)$*,* $m(\psi)$*,* $v(\psi)$ *and* $\mathbb{V}(\psi)$ *are genetic invariants. There exists a unique Fein field* $\mathrm{Fein}(\psi)$*. Furthermore,* $\mathrm{Fein}(\psi)$ *is a genetic invariant. The* $\mathrm{Aut}(\mathbb{V}(\psi))$*-set isomorphism class of vertex set* $\mathrm{Vtx}(\psi)$*, the Frobenius–Schur type* $\Delta(\psi)$ *and the genotype* $\mathrm{Typ}(\psi)$ *are genetic invariants.*

Proof. Obviously, $\mathrm{Typ}(\psi)$ is a global invariant. By the Field-Changing Theorem 3.5, the genetic subquotients for ψ coincide with the genetic subquotients for $\psi_{\mathbb{Q}}$. Therefore $\mathrm{Typ}(\psi)$ is a quasiconjugacy invariant. Given a subgroup $L \leq G$ and a $\mathbb{K}L$-irrep θ from which ψ is tightly induced, then, by Remark 3.3, every genetic subquotient for θ is a genetic subquotient for ψ. Therefore $\mathrm{Typ}(\psi)$ is a tight induction invariant. We have shown that $\mathrm{Typ}(\psi)$ is a genetic invariant.

Let us write $[\mathrm{End}_{\mathbb{Q}}]$ to denote the isomorphism class of the ring $\mathrm{End}_{\mathbb{Q}} = \mathrm{End}_{\mathbb{Q}G}(\psi_{\mathbb{Q}})$. As we already noted in Section 2, $[\mathrm{End}_{\mathbb{Q}}]$ is a quasiconjugacy global invariant. By the definition of shallow induction, $[\mathrm{End}_{\mathbb{Q}}]$ is a tight induction invariant. So $[\mathrm{End}_{\mathbb{Q}}]$ is a genetic invariant.

The $\mathbb{J}$-algebra $\mathbb{J} \otimes_{\mathbb{Q}} \mathrm{End}_{\mathbb{Q}}$ is isomorphic to a direct sum of $v_{\mathbb{J}}(\psi)$ copies of the ring of $m_{\mathbb{J}}(\psi) \times m_{\mathbb{J}}(\psi)$ matrices over the $\mathbb{J}$-algebra $\mathrm{End}_{\mathbb{J}} = \mathrm{End}_{\mathbb{J}G}(\mathbb{J}\psi_{\mathbb{Q}})$. So $[\mathrm{End}_{\mathbb{Q}}]$ determines $v_{\mathbb{J}}(\psi)$ and $m_{\mathbb{J}}(\psi)$. Furthermore, $[\mathrm{End}_{\mathbb{Q}}]$ determines the isomorphism class of $\mathrm{End}_{\mathbb{J}}$ and, in particular, the isomorphism class of $\mathrm{End}_{\mathbb{R}} = \Delta_{\psi}$. The $\mathbb{L}$-algebra $\mathbb{L} \otimes_{\mathbb{J}} \mathrm{End}_{\mathbb{J}}$ is isomorphic to a direct sum of $v_{\mathbb{J}}^{\mathbb{L}}(\psi)$ copies of the ring of $m_{\mathbb{J}}^{\mathbb{L}}(\psi) \times m_{\mathbb{J}}^{\mathbb{L}}(\psi)$ matrices over $\mathrm{End}_{\mathbb{L}}$. So $[\mathrm{End}_{\mathbb{Q}}]$ determines $v_{\mathbb{J}}^{\mathbb{L}}(\psi)$ and $m_{\mathbb{J}}^{\mathbb{L}}(\psi)$. We have $\mathbb{V}_{\mathbb{J}}(\psi) \cong Z(\mathrm{End}_{\mathbb{J}})$, so $[\mathrm{End}_{\mathbb{Q}}]$ determines $\mathbb{V}_{\mathbb{J}}(\psi)$. The splitting fields for $\psi_{\mathbb{J}}$ are precisely the splitting fields for $\mathrm{End}_{\mathbb{J}}$. So $[\mathrm{End}_{\mathbb{Q}}]$ determines the class of minimal splitting fields for $\psi_{\mathbb{J}}$. It follows that $[\mathrm{End}_{\mathbb{Q}}]$ determines the $n(\psi)$ and the class of Fein fields for ψ. As $\mathrm{Aut}(\mathbb{V}(\psi))$-sets, $\mathrm{Vtx}(\psi)$ is isomorphic to the set of Wedderburn components of the semisimple ring $\mathbb{V}(\psi) \otimes_{\mathbb{Q}} \mathrm{End}_{\mathbb{Q}}$. So $[\mathrm{End}_{\mathbb{Q}}]$ determines $\mathrm{Vtx}(\psi)$. With the exception of $\mathrm{Typ}(\psi)$, all the specified invariants are thus determined by the genetic invariant $[\mathrm{End}_{\mathbb{Q}}]$, hence they are genetic invariants. By the way, an easier way to see the genetic invariance of $\mathrm{Vtx}(\psi)$ is to observe that the quasiconjugacy global invariance is obvious, while the tight induction invariance is immediate from condition (b) in the Narrow Lemma 3.2.

It remains only to demonstrate the existence and uniqueness of the Fein field. Let H/K be a genetic subquotient of ψ and let ϕ be the germ of ψ at H/K. Since the class of Fein fields is a genetic invariant, the Fein fields of ψ coincide with the Fein fields of ϕ. Replacing ψ with ϕ, we reduce to the case where G is Roquette and ψ is faithful. If $G = C_1$ or $G = C_2$, then the unique $\mathbb{Q}$-relative Fein field is $\mathrm{Fein}(\psi) = \mathbb{Q}$. When $|G| \geq 3$, the existence and uniqueness of $\mathrm{Fein}(\psi)$ was already shown in Lemmas 5.1, 5.2, 5.3, 5.4, 5.5. □

The argument in the last paragraph of the proof can be abstracted in the form of the following remark.

Remark 5.8. *Let ψ be a $\mathbb{K}G$-irrep. Let $\mathbb{J}$ be a field with characteristic zero, and let ϕ be a faithful $\mathbb{J}\mathrm{Typ}(\psi)$-irrep. Let $\mathcal{I}$ be an invariant defined on characteristic zero irreps of finite p-groups. If $\mathcal{I}$ is a genetic invariant, then $\mathcal{I}(\psi) = \mathcal{I}(\phi)$.*

Thus, any genetic invariant is determined by the genotype. Conversely, the latest theorem tells us that the genotype is a genetic invariant. The following corollary is a restatement of those two conclusions.

Corollary 5.9. *For irreps of finite p-groups over a field with characteristic zero, the genetic invariants are precisely the isomorphism invariants of the genotype.*

The field $\mathrm{Fein}(\psi)$ need not be the only minimal splitting field for ψ contained in $\mathbb{Q}_{|G|}$. Lemma 5.5 shows that every quaternion 2-group is a counter-example. For a complex irrep χ of an arbitrary finite group F, the splitting field $\mathbb{Q}_{|G|}$ need not contain a minimal splitting field for χ. Fein [8] gave a counter-example where $|F|$ has precisely three prime factors. In the same paper, he showed that, if $|F|$ has precisely two prime factors and if χ has Schur index $m(\chi) \geq 3$, then $\mathbb{Q}_n$ contains a minimal splitting field, where n is the exponent of G. However, as we are about to show, the condition $m(\chi) \geq 3$ always fails when F is a p-group. The line of argument by which we arrive at the following celebrated result is due to Roquette [14], but it is worth assimilating into our account because it is a paradigm for the genetic reduction technique.

Theorem 5.10. (Schilling's Theorem) *Given a $\mathbb{K}G$-irrep ψ and a field extension $\mathbb{L}/\mathbb{J}$ with characteristic zero, then $m_{\mathbb{J}}^{\mathbb{L}}(\psi) \leq 2$. If $m_{\mathbb{J}}^{\mathbb{L}}(\psi) = 2$ then $\Delta(\psi) = \mathbb{H}$. If $\Delta(\psi) = \mathbb{H}$, then $m(\psi) = 2$.*

Proof. By the latest theorem and the subsequent remark, we may assume that G is Roquette and that ψ is faithful. From the definition of the relative Schur index, $m_{\mathbb{Q}}^{\mathbb{L}}(\psi) = m_{\mathbb{Q}}^{\mathbb{J}}(\psi) m_{\mathbb{J}}^{\mathbb{L}}(\psi)$. So $m_{\mathbb{J}}^{\mathbb{L}}(\psi) \leq m_{\mathbb{Q}}^{\mathbb{L}}(\psi) \leq m(\psi)$. It suffices to show that $m(\psi) \leq 2$ with equality if and only if

$\Delta(\psi) = \mathbb{H}$. Applying Lemmas 5.1, 5.2, 5.3, 5.4, and attending separately to the degenerate case $|G| \leq 2$, we deduce that if G is cyclic, dihedral or semidihedral, then $m(\psi) = 1$ and $\Delta(\psi) \neq \mathbb{H}$. If G is quaternion then, by Lemma 5.5, $m(\psi) = 2$ and $\Delta(\psi) = \mathbb{H}$. □

The next result is probably of no technical interest, but it does at least indicate why we call $\mathbb{V}(\psi)$ the vertex field. However, the analogous assertion can fail for the relative vertex field: if G is a quaternion 2-group then $\mathbb{V}_{\mathbb{R}}(\psi) = \mathbb{R}$, but the unique minimal splitting field for ψ is $\mathbb{C}$.

Proposition 5.11. *Let ψ be a $\mathbb{K}G$-irrep. Partially ordering isomorphism classes of fields by embedding, then (the isomorphism class of) $\mathbb{V}(\psi)$ is the unique maximal field that embeds in all the minimal splitting fields of ψ.*

Proof. By the latest theorem and remark, we may assume that G is Roquette and that ψ is faithful. The assertion is now clear from Lemmas 5.1, 5.2, 5.3, 5.4, 5.5. □

We shall end this movement by showing that the genotype $\mathrm{Typ}(\psi)$ of a non-trivial $\mathbb{K}G$-irrep ψ is determined by the ring $\mathrm{End}_{\mathbb{Q}G}(\psi_{\mathbb{Q}})$, and the genotype is also determined by the class of minimal splitting fields for $\psi_{\mathbb{Q}}$. Note that, aside from the genotype itself, none of the genetic invariants listed in Theorem 5.7 can be used to distinguish between genotype C_1 and genotype C_2. But those two genotypes can be distinguished very easily: a Frobenius reciprocity argument shows that $\mathrm{Typ}(\psi) = C_1$ if and only if ψ is the trivial $\mathbb{K}G$-irrep.

The following corollary relies on the Genotype Theorem 1.1. Indeed, it relies on Theorem 5.7. Although we did not mention the Genotype Theorem in the above proof of Theorem 5.7, we implicitly used the Genotype Theorem because our argument involved $\mathrm{Typ}(\psi)$, whose existence and uniqueness is guaranteed by the Genotype Theorem. However, the reasoning that has led us to the following corollary makes essential use only of the existence, not the uniqueness. The existence of $\mathrm{Typ}(\psi)$ is captured in Theorem 4.1, which was proved by a direct argument in Section 4. So, with the following corollary, we complete a direct proof of the Genotype Theorem, avoiding the reduction to the special case $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{C}$.

Corollary 5.12. *Let ψ be a non-trivial $\mathbb{K}G$-irrep. Let ψ' be a non-trivial $\mathbb{K}G'$-irrep, where G' is a finite p'-group and p' is a prime. Then the following conditions are equivalent.*

(a) $\mathrm{Typ}(\psi) = \mathrm{Typ}(\psi')$.

(b) $\mathrm{End}_{\mathbb{Q}G}(\psi_{\mathbb{Q}}) \cong \mathrm{End}_{\mathbb{Q}G'}(\psi'_{\mathbb{Q}})$.

(c) *The minimal splitting fields for ψ coincide with the minimal splitting fields for ψ'.*

Proof. Since the endomorphism ring $\mathrm{End}_{\mathbb{Q}G}(\psi_{\mathbb{Q}})$ is a genetic invariant, it is isomorphic to the endomorphism ring of the faithful rational irrep of $\mathrm{Typ}(\psi)$. So (a) implies (b). The minimal splitting fields for ψ are precisely the minimal splitting fields for $\mathrm{End}_{\mathbb{Q}G}(\psi_{\mathbb{Q}})$. So (b) implies (c). Suppose that (c) holds. To deduce (a), the latest theorem and remark allow us to assume that G and G' are Roquette. If $\mathbb{Q}$ is a splitting field for ψ and ψ', then $\mathrm{Typ}(\psi) = C_2 = \mathrm{Typ}(\psi')$. Otherwise, the equality of the two genotypes follows from the first five lemmas in this section. □

Finally, we are ready to present the synthesis of the material in the previous two movements. Corollary 5.12 is unlikely to be of much use towards evaluating the genotype of an explicitly given irrep. The following theorem can be applied first to evaluate the genotype from more easily ascertained genetic invariants. The genotype having been evaluated, the theorem can be applied again to evaluate other genetic invariants. (The above proof of Schilling's Theorem can be cast in that form. Anyway, we are not suggesting that anyone would actually wish to evaluate genetic invariants for numerically specified irreps. It can be argued that, in pure mathematics no less than in the other sciences, a sufficient criterion for meaningful content should be only that the material could be applied efficiently and effectively to some natural class of problems; without requiring that there be any demand for the solutions to those problems.)

Theorem 5.13. *Let ψ be a $\mathbb{K}G$-irrep and let $v = v(\psi)$. First suppose that $v = 1$. Then precisely one of the following three conditions holds.*

(a) $\mathrm{Typ}(\psi) = C_1$ *and ψ is the trivial $\mathbb{K}G$-irrep.*

(b) $\mathrm{Typ}(\psi) = C_2$ *and ψ is non-trivial, affordable over $\mathbb{Q}$ and absolutely irreducible. In particular, the Schur index is $m(\psi) = 1$ and the Frobenius–Schur type is $\Delta(\psi) = \mathbb{R}$.*

(c) $\mathrm{Typ}(\psi) = Q_8$ *and $m(\psi) = 2$ and $\Delta(\psi) = \mathbb{H}$.*

Now suppose that p is odd and $v \neq 1$. Then the exponent $n = n(\psi)$ is a power of p and $v = n(1 - 1/p)$. Also, $m(\psi) = 1$ and $\Delta(\psi) = \mathbb{C}$. The unique minimal splitting field for ψ is the field $\mathrm{Fein}(\psi) = \mathbb{V}(\psi) = \mathbb{Q}_n$*. The vertex set* $\mathrm{Vtx}(\psi)$ *is free and transitive as an* $\mathrm{Aut}(\mathbb{Q}_n)$*-set.*

Now suppose that $p = 2$ and $v \neq 1$. Then v is a power of 2 and precisely one of the following conditions holds.

(C) $\mathrm{Typ}(\psi) = C_{2v}$ *and $n(\psi) = 2v$ and $m(\psi) = 1$ and $\Delta(\psi) = \mathbb{R}$. The unique minimal splitting field for ψ is the field* $\mathrm{Fein}(\psi) = \mathbb{V}(\psi) = \mathbb{Q}_{2v}$*. As a transitive* $\mathrm{Aut}(\mathbb{Q}_{4v})$*-set,* $\mathrm{Vtx}(\psi)$ *has point-stabilizer subgroup $\langle \gamma \rangle$.*

(D) $\mathrm{Typ}(\psi) = D_{8v}$ *and $n(\psi) = 4v$ and $m(\psi) = 1$ and $\Delta(\psi) = \mathbb{R}$. The unique minimal splitting field for ψ is the field* $\mathrm{Fein}(\psi) = \mathbb{V}(\psi) = \mathbb{Q}^{\mathbb{R}}_{4v}$*. As a transitive* $\mathrm{Aut}(\mathbb{Q}_{4v})$*-set,* $\mathrm{Vtx}(\psi)$ *has point-stabilizer subgroup $\langle \beta \rangle$.*

(S) $\mathrm{Typ}(\psi) = \mathrm{SD}_{8v}$ *and $n(\psi) = 4v$ and $m(\psi) = 1$ and $\Delta(\psi) = \mathbb{C}$. The unique minimal splitting field for ψ is the field* $\mathrm{Fein}(\psi) = \mathbb{V}(\psi) = \mathbb{Q}^{\mathbb{I}}_{4v}$*. As a transitive* $\mathrm{Aut}(\mathbb{Q}_{4v})$*-set,* $\mathrm{Vtx}(\psi)$ *has point-stabilizer subgroup $\langle \delta \rangle$.*

(Q) $\mathrm{Typ}(\psi) = Q_{8v}$ *and $n(\psi) = 4v$ and $m(\psi) = 2$ and $\Delta(\psi) = \mathbb{H}$. Two non-isomorphic splitting fields for ψ are $\mathbb{Q}_{4v}$ and $\mathbb{Q}^{\mathbb{I}}_{8v}$. Also,* $\mathrm{Fein}(\psi) = \mathbb{Q}_{4v}$ *and* $\mathbb{V}(\psi) = \mathbb{Q}^{\mathbb{R}}_{4v}$*. As a transitive* $\mathrm{Aut}(\mathbb{Q}_{4v})$*-set,* $\mathrm{Vtx}(\psi)$ *has point-stabilizer subgroup $\langle \beta \rangle$.*

Proof. The case $|\mathrm{Typ}(\psi)| \leq 2$ is easy. The rest follows from the first eight results in this section. □

When $p = 2 \leq v$, one routine for calculating the genotype is to find the values of v and $\mathrm{fs}(\psi)$. If $\mathrm{fs}(\psi) = 0$ then the possible genotypes C_{2v} and SD_{8v} can be distinguished using the fact that, in the former case, the involution fixing ψ is γ while, in the latter case, the involution fixing ψ is δ. Another routine is to evaluate $\mathbb{V}(\psi)$ and, if necessary, $\mathrm{fs}(\psi)$.

To reinforce the point, let us return, once again, to the real irrep χ of the group $\mathrm{DD} = \mathrm{DD}_{2^n} = \mathrm{DD}_{16u}$. We evaluated $\mathrm{Typ}(\psi)$ already at the end of Section 4, but let us now do it more swiftly. Using part (3) of Lemma 2.5, we see that $v(\psi) = u/2$. By considering the partition

$$\mathrm{DD} = A \cup (D_{8u} - A) \cup (\mathrm{Mod}_{8u} - A) \cup (\mathrm{SD}_{8u} - A)$$

we see that $\mathrm{fs}(\chi) = 1$ and $\Delta(\chi) = \mathbb{R}$. We recover the conclusion that $\mathrm{Typ}(\chi) = D_{2^{n-2}}$ if $n \geq 6$ while $\mathrm{Typ}(\chi) = C_2$ if $n = 5$.

For another example, still with $n \geq 5$, suppose that G is the smash product $C_4 * D_{2^{n-1}}$, which has order 2^n. Each faithful complex irrep of the subgroup $1 * D_{2^{n-1}}$ extends to two complex conjugate irreps of G. So there are precisely 2^{n-3} faithful $\mathbb{C}G$-irreps, and they comprise a single Galois conjugacy class. Let ψ be a faithful $\mathbb{K}G$-irrep. Then $v(\psi) = 2^{n-3}$. The two generators of $C_4 * 1$ act on $\psi_\mathbb{C}$ as scalar multiplication by $\pm i$, and the faithful complex irreps of $1 * D_{2^{n-1}}$ have vertex field $\mathbb{Q}^\mathbb{R}_{2^{n-2}}$, hence

$$\mathbb{V}(\psi) = \mathbb{Q}^\mathbb{R}_{2^{n-2}}[i] = \mathbb{Q}_{2^{n-2}} = \mathbb{Q}_{2v(\psi)} \, .$$

Therefore, $\mathrm{Typ}(\psi) = C_{2^{n-2}}$. At the beginning of Section 2, we noted that the unique faithful $\mathbb{Q}C_4 * D_{16}$-irrep is induced from a C_4 subquotient and also from a C_8 subquotient; we have made some progress since then, and we can now announce that, actually, the genotype is C_8.

6 Counting Galois conjugacy classes of irreps

We shall be correlating some results of tom Dieck [7, III.5.9] and Bouc [4, 8.5, 8.7]. They were concerned with the cases where $\mathbb{K}$ is $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. Our generalizations to the case of arbitrary $\mathbb{K}$ are slender, although we should point out that the construction of $\overline{R}(\mathbb{K}G)$, below, does rely on the notion of Galois conjugacy that was established in Section 2. What is of more interest is that we shall be providing quicker proofs, making use of the fact that the genetic theory applies directly to the general case.

Remark 6.1. *Let R be a Roquette p-group, and let $k_R(G)$ denote the number of Galois conjugacy classes of $\mathbb{K}G$-irreps with genotype R. Then $k_R(G)$ is independent of $\mathbb{K}$.*

The remark is immediate from the Field-Changing Theorem 3.5. We mention that $k_R(G)$ is a global quasiconjugacy invariant. Letting $k_*(G) = \sum_R k_R(G)$, where R runs over all the Roquette p-groups, then $k_*(G)$ is the number of Galois conjugacy classes of $\mathbb{K}G$-irreps, in other words, the number of $\mathbb{Q}G$-irreps, we mean to say, the number of conjugacy classes of cyclic subgroups of G.

Recall that a **superclass** function for G is a $\mathbb{Z}$-valued function f on the set of subgroups of G such that f is constant on each conjugacy class of subgroups. The **superclass ring** of G, denoted $C(G)$, is understood to be the additive group consisting of the superclass functions on G. The **representation ring** $R(\mathbb{K}G)$, also called the **character ring**, is understood to be the group of virtual $\mathbb{K}G$-reps (the universal abelian group associated with the semigroup of $\mathbb{K}G$-reps). Of course, in many well-known applications, $C(G)$ and $R(\mathbb{K}G)$ are assigned all sorts of further structures (in particular, they are rings) but those further structures are irrelevant to our concerns. We shall be regarding $C(G)$ and $R(\mathbb{K}G)$ merely as free abelian groups.

We define the **tom Dieck map**

$$\mathrm{Die}^\mathbb{K}_G \, : \, R(\mathbb{K}G) \to C(G)$$

to be the linear map such that, given a $\mathbb{K}G$-rep ξ and a subgroup $H \leq G$, then the value of $\mathrm{Die}^\mathbb{K}_G(\xi)$ at H is equal to the multiplicity of the trivial $\mathbb{K}H$-irrep in the restriction $\mathrm{res}^G_H(\xi)$. Thus, treating ξ as a $\mathbb{K}G$-module and writing its H-fixed subspace as ξ^H, we have $\mathrm{Die}^\mathbb{K}_G(\xi)(H) = \dim_\mathbb{K}(\xi^H)$. Let $I(\mathbb{K}G)$ be the subgroup of $R(\mathbb{K}G)$ generated by the elements having the form

$\xi - \xi'$, where ξ and ξ' are Galois conjugate $\mathbb{K}G$-reps. We write $\overline{\xi} = \xi + I(\mathbb{K}G)$ as an element of the quotient group $\overline{R}(\mathbb{K}G) = R(\mathbb{K}G)/I(\mathbb{K}G)$. Since $\mathrm{Die}_G^{\mathbb{K}}$ annihilates $I(\mathbb{K}G)$, we can define another **tom Dieck map**

$$\overline{\mathrm{Die}}_G^{\mathbb{K}} : \overline{R}(\mathbb{K}G) \to C(G)$$

such that $\overline{\mathrm{Die}}_G^{\mathbb{K}}(\overline{\xi}) = \mathrm{Die}_G^{\mathbb{K}}(\xi)$.

Letting ψ_1, ..., ψ_r be a set of representatives of the Galois conjugacy classes of $\mathbb{K}G$-irreps, then $\{\overline{\psi}_1, ..., \overline{\psi}_r\}$ is a $\mathbb{Z}$-basis for $\overline{R}(\mathbb{K}G)$, and $\{\overline{(\psi_1)_{\mathbb{Q}}}, ..., \overline{(\psi_r)_{\mathbb{Q}}}\}$ is a $\mathbb{Z}$-basis for $\overline{R}(\mathbb{Q}G)$. We mention that there is a isomorphism $\overline{R}(\mathbb{K}G) \to \overline{R}(\mathbb{Q}G)$ such that $\overline{\psi}_j \mapsto \overline{(\psi_j)_{\mathbb{Q}}}$. But the isomorphism does not commute with the tom Dieck maps $\overline{\mathrm{Die}}_G^{\mathbb{K}}$ and $\overline{\mathrm{Die}}_G^{\mathbb{Q}}$.

The next two results are due to tom Dieck [7, III.5.9, III.5.17]. Our slight embellishment is to extend to the case where $\mathbb{K}$ is arbitrary. Although the proofs presented below are different from tom Dieck's, the ideas are implicit in [7, III.5].

Theorem 6.2. (tom Dieck) *The tom Dieck map* $\overline{\mathrm{Die}}_G^{\mathbb{K}}$ *is injective, and its image is a free abelian group whose rank is* $k_*(G)$.

Proof. By Theorem 2.2, we may assume that $\mathbb{K} = \mathbb{C}$. Suppose that $\overline{\mathrm{Die}}_G^{\mathbb{C}}$ is not injective. Then

$$a_1 \dim_{\mathbb{C}}(\psi_1^H) + ... + a_r \dim_{\mathbb{C}}(\psi_r^H) = \mathrm{Die}_G^{\mathbb{C}}(a_1\psi_1 + ...a_r\psi_r) = 0$$

where ψ_1, ..., ψ_r are mutually Galois non-conjugate $\mathbb{C}G$-irreps and each a_j is a non-zero integer. First consider the case where some ψ_j is non-faithful. Then, without loss of generality, there is an integer $s \leq r$ and a non-trivial normal subgroup K of G such that the kernels of ψ_1, ..., ψ_s all contain K while the kernels of ψ_{s+1}, ..., ψ_r do not contain K. When $s < j \leq r$, Clifford theory yields $\dim_{\mathbb{C}}(\psi_j^K) = 0$ and, perforce, $\dim_{\mathbb{C}}(\psi_j^H) = 0$ for all intermediate subgroups $K \leq H \leq G$. Replacing G with G/K, we obtain a contradiction by insisting that $|G|$ was minimal. Now consider the case where all the ψ_j are faithful and G is not Roquette. Let E and T be as in Lemma 4.2. The proof of that lemma shows that each $\mathrm{res}_T^G(\psi_j) = \psi_{j,1} + ... + \psi_{j,p}$ as a sum of mutually Galois non-conjugate $\mathbb{C}T$-irreps. But $\psi_j = \mathrm{ind}_T^G(\psi_{j,i})$ and it follows that, as j and i run over the ranges $1 \leq j \leq r$ and $1 \leq i \leq p$, the $\mathbb{C}T$-irreps $\psi_{j,i}$ are mutually Galois non-conjugate. Replacing G with T, we again obtain a contradiction by induction on $|G|$. We have reduced to the case where G is Roquette and ψ_1, ..., ψ_r are faithful. By Corollary 2.8, $r = 1$. Absurdly, we deduce that $\dim_{\mathbb{C}}(\psi_1^H) = 0$ for all subgroups $H \leq G$. □

We write mod_2 to indicate reduction modulo 2: for a free abelian group A, we write $\mathrm{mod}_2(A) = (\mathbb{Z}/2) \otimes_{\mathbb{Z}} A$; we write mod_2 for canonical epimorphism $A \to \mathrm{mod}_2(A)$. The composite maps $\mathrm{mod}_2 \mathrm{Die}_G^{\mathbb{K}} : R(\mathbb{K}G) \to \mathrm{mod}_2(C(G))$ and $\mathrm{mod}_2 \overline{\mathrm{Die}}_G^{\mathbb{K}} : \overline{R}(\mathbb{K}G) \to \mathrm{mod}_2(C(G))$ are still called **tom Dieck maps**.

Theorem 6.3. (tom Dieck) *Suppose that* $p = 2$. *Then the image* $\mathrm{Im}(\mathrm{mod}_2 \overline{\mathrm{Die}}_G^{\mathbb{K}})$ *is an elementary abelian* 2*-group whose rank is the number of Galois conjugacy classes of* $\mathbb{K}G$*-abirreps with cyclic, dihedral or semidihedral genotype. These are the abirreps with Frobenius–Schur type* $\mathbb{R}$ *or* $\mathbb{C}$.

Proof. The rider will follow from the main part together with Theorem 5.13. For any $\mathbb{K}G$-irrep ψ, the order $v(\psi)$ and the Schur multiplier $m(\psi)$ are powers of 2. Therefore $\mathrm{mod}_2 \mathrm{Die}_G^{\mathbb{K}}$ annihilates any $\mathbb{K}G$-irrep that is not absolutely irreducible. So, if the required conclusion holds for the algebraic closure of $\mathbb{K}$, then it will hold for $\mathbb{K}$. Therefore, we may assume that $\mathbb{K} = \mathbb{C}$.

Let ψ_1, ..., ψ_r be mutually Galois non-conjugate $\mathbb{C}G$-irreps with non-quaternion genotypes. We claim that the linearly independent elements $\overline{\psi}_1$, ..., $\overline{\psi}_r$ of $\overline{R}(\mathbb{C}G)$ are sent by $\mathrm{mod}_2 \, \overline{\mathrm{Die}}{}_G^{\mathbb{K}}$ to linearly independent elements of $\mathrm{mod}_2(C(G))$. Assuming otherwise, and taking r to be minimal, then

$$\overline{\mathrm{Die}}{}_G^{\mathbb{K}}(\overline{\psi}_1) + ... + \mathrm{Die}_G^{\mathbb{K}}(\overline{\psi}_r) = 0.$$

That is to say, the dimension of $\psi_1^H \oplus ... \oplus \psi_r^H$ is even for all $H \leq G$. Arguing as in the proof of the previous theorem, we reduce to the case where $r = 1$ and ψ_1 is faithful and G is a non-quaternion Roquette 2-group. Write $\psi = \psi_1$. If G is cyclic, then the subspace fixed by the trivial group has dimension $\dim_{\mathbb{C}}(\psi^1) = 1$, and this is a contradiction. Supposing that G is dihedral, and writing $G = \langle a, b \rangle$ as in the standard presentation, then $\dim_{\mathbb{C}}(\psi^{\langle b \rangle}) = 1$, which is a contradiction. In the semidihedral case we obtain a contradiction using the standard generator d instead of b. Any which way, we arrive at a contradiction. The claim is established.

Now let G be any 2-group and let ψ be a $\mathbb{C}G$-irrep having genotype Q_{2^m} with $m \geq 3$. The argument will be complete when we have shown that $\overline{\mathrm{Die}}{}_G^{\mathbb{C}}(\overline{\psi}) = 0$. In other words, it remains only to show that ψ^H has even dimension for all $H \leq G$. Consider the case where $G = Q_{2^m}$. The center $Z = Z(G)$ has order 2, and it is contained in every non-trivial subgroup of G. There are no non-zero Z-fixed points in ψ, hence $\dim_{\mathbb{C}}(\psi^H)$ is 2 or 0, depending on whether H is trivial or non-trivial, respectively. Either way, $\dim_{\mathbb{C}}(\psi^H)$ is even. Return now to the case where G is arbitrary. Let L/M be a genetic subquotient for ψ and let ϕ be the germ of ψ at L/M. Of course, $L/M \cong Q_{2^m}$. By Frobenius reciprocity and Mackey decomposition,

$$\dim_{\mathbb{C}}(\psi^H) = \langle 1 | \mathrm{res}_H^G \, \mathrm{ind}_L^G(\psi) \rangle = \sum_{LgH \subseteq G} \langle 1 | \mathrm{res}_{L \cap {}^g H}^L(\psi) \rangle = \sum_{LgH \subseteq G} \dim_{\mathbb{C}}(\phi^{L \cap {}^g H}).$$

From our discussion of the case $G = Q_{2^m}$, we see that each term of the sum is 0 or 2. So $\dim_{\mathbb{C}}(\psi^H)$ is even. □

We now explain how, in the special case where $\mathbb{K}$ is a subfield of $\mathbb{R}$, the group $\mathrm{mod}_2(C(G))$ can be replaced with the unit group $B(G)^\times$ of the Burnside ring $B(G)$. This will lead to a new proof of a theorem of Bouc. We sketch the prerequisite constructions, referring to Yoshida [19] and Yalçın [17] for details (we employ much the same notation as the latter). Given a G-set X, we write $[X]$ to denote the isomorphism class of X as an element of $B(G)$. Recall that the species of $B(G)$ have the form $s_H : B(G) \ni [X] \mapsto |X^H| \in \mathbb{Z}$, and two species s_H and $s_{H'}$ are equal if and only if H and H' are G-conjugate. Furthermore, an element x of $B(G)$ is determined by the superclass function $H \mapsto s_H(x)$. Note that x is a unit if and only if $s_H(x) = \pm 1$ for all H. Therefore, $B(G)^\times$ is an elementary abelian 2-group.

Given an integer c (possibly given only up to congruence modulo 2), we write $\mathrm{par}(c) = (-1)^c$. Another theorem of tom Dieck [6, 5.5.9] asserts that, given any $\mathbb{R}G$-rep ξ, then there is an element $\mathrm{die}_G^{\mathbb{R}}(\xi) \in B(G)^\times$ such that $s_H(\mathrm{die}_G^{\mathbb{R}}(\xi)) = \mathrm{par}(\dim_{\mathbb{R}}(\xi^H))$. Hence, when $\mathbb{K}$ is a subfield of $\mathbb{R}$, there is a linear map

$$\mathrm{die}_G^{\mathbb{K}} \; : \; R(\mathbb{K}G) \to B(G)^\times$$

such that, again, $s_H(\mathrm{die}_G^{\mathbb{K}}(\xi)) = \mathrm{par}(\dim_{\mathbb{K}}(\xi^H))$. Since $\mathrm{die}_G^{\mathbb{K}}$ annihilates the ideal $I(\mathbb{K}G)$, it gives rise to a linear map

$$\overline{\mathrm{die}}{}_G^{\mathbb{K}} \; : \; \overline{R}(\mathbb{K}G) \to B(G)^\times \, .$$

The maps $\mathrm{die}_G^{\mathbb{K}}$ and $\overline{\mathrm{die}}{}_G^{\mathbb{K}}$ are called **tom Dieck maps** because, as we shall see in a moment, they are essentially the same as the maps $\mathrm{Die}_G^{\mathbb{K}}$ and $\overline{\mathrm{Die}}{}_G^{\mathbb{K}}$. But let us emphasize that $\mathrm{die}_G^{\mathbb{K}}$ and $\overline{\mathrm{die}}{}_G^{\mathbb{K}}$ are defined only when $\mathbb{K} \leq \mathbb{R}$.

The following result can be quickly obtained from the special case $\mathbb{K} = \mathbb{R}$, which is equivalent to Bouc [4, 8.5]. See Corollary 6.6. We give a different proof.

Theorem 6.4. (Bouc) *Suppose that* $\mathbb{K} \leq \mathbb{R}$. *Then the image* $\mathrm{Im}(\overline{\mathrm{die}}_G^{\mathbb{K}} : \overline{R}(\mathbb{K}G) \to B(G)^\times)$ *is an elementary abelian* 2*-group whose rank is the number of Galois conjugacy classes of absolutely irreducible* $\mathbb{K}G$*-irreps.*

Proof. When p is odd, the assertion is virtually trivial. Indeed, for any group of odd order, the trivial irrep is the unique real abirrep, and meanwhile, tom Dieck [6, 1.5.1] asserts that, for any group of odd order, the unit group of the Burnside ring is isomorphic to C_2.

Suppose that $p = 2$. The tom Dieck maps $\mathrm{mod}_2 \overline{\mathrm{Die}}_G^{\mathbb{K}} : \overline{R}(\mathbb{K}G) \to \mathrm{mod}_2(C(G))$ and $\mathrm{mod}_2 \overline{\mathrm{die}}_G^{\mathbb{K}} : \overline{R}(\mathbb{K}G) \to B(G)^\times$ commute with the monomorphism of elementary abelian 2-groups $B(G)^\times \to \mathrm{mod}_2(C(G))$ which sends a unit u to the superclass function f such that $s_H(u) = \mathrm{par}(f(H))$. By Theorem 6.3, the rank $\mathrm{rk}(\mathrm{Im}(\mathrm{mod}_2 \overline{\mathrm{Die}}_G^{\mathbb{K}})) = \mathrm{rk}(\mathrm{Im}(\mathrm{mod}_2 \overline{\mathrm{die}}_G^{\mathbb{K}}))$ is the number of Galois conjugacy classes of $\mathbb{K}G$-abirreps with Frobenius–Schur type $\mathbb{R}$ or $\mathbb{C}$. The hypothesis on $\mathbb{K}$ ensures that all the $\mathbb{K}G$-abirreps have type $\mathbb{R}$. □

In order to recover the two special cases stated in Bouc [4, 8.5, 8.7], we need the following result, which was obtained by Tornehave [15] using topological methods. Another proof was given by Yalçın [17] using algebraic methods. Actually, Yalçın reduced to the case of a Roquette 2-group, and his paper is another application of the genetic reduction technique. Note that the case of odd p is trivial.

Theorem 6.5. (Tornehave) *The tom Dieck map* $\overline{\mathrm{die}}_G^{\mathbb{R}} : \overline{R}(\mathbb{R}G) \to B(G)^\times$ *is surjective.*

From the latest two results, we recover the following corollary, which was obtained by Bouc [4, 8.5] using a filtration of $B(—)^\times$ as a biset functor.

Corollary 6.6. (Bouc) *The number of Galois conjugacy classes of* $\mathbb{R}G$*-abirreps is the rank of* $B(G)^\times$ *as an elementary abelian* 2*-group.*

It is worth pointing out how the general version of Bouc's Theorem 6.4 covers another of his results, [4, 8.7], which we shall present as the next corollary. Let lin_G denote the linearization map $B(G) \to R(\mathbb{K}G)$. We mean to say, $\mathrm{lin}_G[X]$ is the permutation $\mathbb{K}G$-rep associated with the G-set X. Let $\exp_G$ denote the exponential map $B(G) \to B(G)^\times$. We mean to say, $s_H(\exp_G[X])$ is the number of H-orbits in X. To see that this condition really does determine a unit $\exp_G[X] \in B(G)^\times$, observe that we have a commutative triangle $\exp_G = \mathrm{die}_G^{\mathbb{K}} \mathrm{lin}_G$.

Corollary 6.7. (Bouc) *Suppose that* $p = 2$. *Then the number of* $\mathbb{Q}G$*-abirreps is* $\mathrm{rk}(\mathrm{Im}(\exp_G))$. *Furthermore,* $\exp_G$ *is surjective if and only if there are no* $\mathbb{Q}G$*-irreps with dihedral genotype.*

Proof. The first part follows from Theorem 6.4 together with the Ritter–Segal Theorem, which asserts that the linearization map $\mathrm{lin}_G^{\mathbb{Q}} : B(G) \to R(\mathbb{Q}G)$ is surjective. By Theorem 5.13, the $\mathbb{Q}G$-abirreps are precisely the $\mathbb{Q}G$-irreps whose genotype is C_1 or C_2. Also, the $\mathbb{R}G$-abirreps are precisely the $\mathbb{R}G$-irreps whose genotype is C_1 or C_2 or dihedral. The rider is now clear. □

We end with a further comment on Corollary 6.6. Embedding $R(\mathbb{R}G)$ in $R(\mathbb{C}G)$ via the tensor product $\mathbb{C} \otimes_{\mathbb{R}} —$, we define an elementary abelian 2-group $\overline{O}(G) = R(\mathbb{R}G)/(R(\mathbb{R}G) \cap I^+(\mathbb{C}G))$. Here, $I^+(\mathbb{C}G)$ is the ideal of $R(\mathbb{C}G)$ generated by the elements having the form $\xi + \xi'$, where ξ and ξ' are Galois conjugate $\mathbb{C}G$-reps. The tom Dieck map $\overline{\mathrm{die}}_G^{\mathbb{R}}$ gives rise to a map $\mathrm{die}_G : \overline{O}(G) \to B(G)^\times$. Theorem 6.5 says that $\overline{\mathrm{die}}_G$ is surjective. Corollary 6.6 says that, in fact, $\overline{\mathrm{die}}_G$ is an isomorphism. Let $B_0(G) = \mathrm{Ker}(\mathrm{lin}_G^{\mathbb{R}})$. Tornehave's topological proof of Theorem 6.5 can be recast in a purely algebraic way which involves maps $\mathrm{rip}_G : B_0(G) \to \overline{O}(G)$ and $\mathrm{torn}_G : B_0(G) \to B(G)^\times$ such that $\mathrm{torn}_G = \overline{\mathrm{die}}_G \, \mathrm{rip}_G$. Such commutative triangles can still be defined when G is replaced by an arbitrary group, although $\overline{\mathrm{die}}_G$ ceases to be an isomorphism in general. The author intends, in a future paper, to discuss some extensions and adaptations of these results of Tornehave. The present paper arose from that work.

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