MATH 104: Thinking Mathematically 2 Spring. Final

20 May 2019, Bilkent University.

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of red pens or very faint pencils is prohibited too. You may take the question sheet home.

Remember to show your working, except where your answers are obvious.

1: 20 marks. In this question, you may use the fact that, for the normal distribution, the probability of a sample lying within one standard deviation of the mean is 68%, and the probabiliy of a sample lying within 2 standard deviations is 95%.

The lengths of adult tree pythons are normally distributed with a mean of 2.5 meters and a standard deviation of 0.25 meters.

(a) A tree python is chosen at random. What is the probability that its length is greater than 3 meters?

(b) Two tree pythons are chosen at random. What is the probability that at least one of them has length less than 2.25 meters?

2: 10 marks. In a particular neighbourhood, the average price of a house is 600 000 euros. The percentage of houses more expensive than 700 000 euros is 2.5. Assuming that the house prices are normally distributed, what is the standard deviation of the house prices?

3: 20 marks. The ballots for four candidates A, B, C, D are as follows. Answer the following questions, showing your working.

number of voters	12	10	8	6
first choice	A	B	C	D
second choice	B	C	D	A
third choice	C	D	A	B
fourth choice	D	A	B	C

(a) Which candidate wins by the plurality method?

(b) Which candidate wins by the Borda count method?

4: 20 marks. For each positive integer n, the graph K_n has n vertices and each vertex has degree $n-1$.

(a) How many edges does K_n have?

(b) For which values of n does K_n have an Euler path?

5: 30 marks. **(a)** The graph of a cube is connected, with 8 vertices, 12 edges, every vertex having the same degree. How many edges need to be added to produce a graph with an Euler circuit?

(b) The graph of a tesseract is connected, with 16 vertices, 32 edges, every vertex having the same degree. What is that degree?

(c) For the graph of a tesseract, how many edges need to be added to produce a graph with an Euler circuit?

Solutions to Final

1: Part (a). The probability of the length being less than 2 meters or greater than 3 meters is 5%. By symmetry, the answer is 2.5%.

Part (b). The probability that a given python has length less than 2.25 meters or more than 2.75 meters is 0.32. So the probability of it having length less than 2.25 meters is 0.16. Since $0.16^2 = 0.0256$, the probability that both pythons have length less than 2.25 meters is 0.0256. So the answer is $0.16 + 0.16 - 0.0256 = 0.2944$.

2: The difference between 700 000 and 600 000 is 2 standard deviations. So the standard deviation is 50 000 euros.

3: Part (a). Eliminating D yields the following table.

number of voters	18	10	8
first choice	A	B	C
second choice	B	C	A
third choice	C	A	B

Eliminating C results in A having a majority of 26 out of 36. So A wins by this method.

Part (b). The Borda counts are as follows:

$$\begin{aligned}
A &: 4.12 + 1.10 + 2.8 + 3.6 = 48 + 10 + 16 + 18 = 92,\\
B &: 3.12 + 4.10 + 1.8 + 2.6 = 36 + 40 + 8 + 12 = 96,\\
C &: 2.12 + 3.10 + 4.6 + 1.6 = 24 + 30 + 32 + 6 = 92,\\
D &: 1.12 + 2.10 + 3.8 + 4.6 = 12 + 20 + 24 = 24 = 80.
\end{aligned}$$

So B wins by this method.

4: Part (a). Any two distinct vertices are linked by an edge. So the number of edges is the number of unordered pairs of distinct vertices. That number is ${}_nC_2 = n(n-1)/2$.

Part (b). The graph K_n has an Euler path if and only if $n = 2$ or n is odd. Indeed, by the Euler Path Theorem, K_n has an Euler circuit if and only if n is odd, while K_n has an Euler path with distinct end-points if and only if $n = 2$.

5: Part (a). By the Euler Path Theorem, we must add 4 edges. For instance, we can add the 4 diagonals.

Part (b). Cutting each edge in half then, since each edge has two-end-points, the number of half-edges is 64. So each vertex has 4 half-edges. In other words, each vertex has degree 4.

Part (c). By the Euler Path Theorem, the graph of a tesseract already has an Euler circuit. The number of edges that need to be added is 0.