Suggested Exercises and Quiz Solutions

MATH 227 Sections 1 and 2, *Introduction to Linear Algebra*, Fall 2018

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Office Hours: Wednesdays, 08:40 – 09:30, Fridays, 16:40 - 17:30. Room SA 129.

Exercises

Most of the questions are taken from the primary textbook: Howard Anton, Chris Rorres, "Elementary Linear Algebra", 11th Edition, 2011 (reprinted 2015).

End of Chapter 1, page 76 onwards, numbers 19 - 37, 51, 52, 102, 125, 127.
End of Chapter 2, page 111, routine: 16 - 20, theoretical: 34, 35, 63, 64.
End of Chapter 3, page 164, routine: 85 - 104, theoretical: 131, 132.

For Midterm 1 preparation, from the exams in the file arch227spr17.pdf:

- All questions from Practice Midterm 1,
- Question 2 from Practice Midterm 2,
- All questions from Midterm 1, 9 March 2017,
- Question 1 from Midterm 2, 28 April 2017.

End of Chapter 4, page 276, routine: 84 - 89, harder or theoretical: 90 - 94.

For Midterm 2 preparation, from the exams in the file arch227spr17.pdf:

- Questions 1, 3, 4, 5 from Practice Midterm 2,
- Questions 2, 3, 4, 5 from Midterm 2, 28 April 2017.

End of Chapter 5, page 325, standard: 3 - 8, 41 - 44, 52 - 54. harder: 34, 55, 56.

An introduction to complex vector spaces through some diagonalization problems:

Question 1: diagonalize the matrix $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

Question 2: For a real number θ, let $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Diagonalize R_θ. Hence evaluate $(R_\theta)^n$ for any integer n. Now give an easier explanation for your evaluation of $(R_\theta)^n$.

For Final preparation: all questions in the Final Exam in the file arch227spr17.pdf.

Quizzes

Quiz 1, Section 1: By Gauss–Jordan elimination, invert $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix}$.

Quiz 1, Section 2: By Gauss–Jordan elimination, invert $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

Quiz 2, Section 1: Find the distance between the point $(3, 4, 4)$ and the plane consisting of the points (x, y, z) such that $2x + 4y - 3z = 6$.

Quiz 2, Section 2: Find the distance between the point $(5, 5, 2)$ and the plane consisting of the points (x, y, z) such that $x + y + 3z = 10$.

Quiz 3, Section 1: Find a basis for the vector space of solutions to the simultaneous equations $x + y + z = 3x + 4y + 5z = 5x + 6y + 7z = 0$. What is the dimension of that vector space?

Quiz 3, Section 2: Find a basis for the vector space of solutions to the simultanous equations $3x + 2y + z = x + 2y + 3z = x + y + z = 0$. What is the dimension of that vector space?

Quiz 4, Section 1: Let S be the subspace of $\mathrm{Mat}_3(\mathbb{R})$ consisting of those matrices A such that $A = A^T$. Find a basis for S. What is the dimension of S?

Quiz 4, Section 2: Let P_6 denote the real vector space of polynomial functions of degree at most 6. Let S be the subspace of P_6 consisting of those f in P_6 such that $f(x) = f(-x)$ for all x. Find a basis for S. What is the dimension of S?

Quiz 5, Section 1: Find all the values λ such that $\det(A - \lambda I) = 0$ where $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

Quiz 5, Section 2: Find all the values z such that there exist nonzero (x, y) satsifying

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = z \begin{bmatrix} x \\ y \end{bmatrix} .$$

Quiz 6, Section 1: Find an invertible matrix P and a diagonal matrix D such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = PDP^{-1} .$$

Quiz 6, Section 2: Find an invertible matrix P and a diagonal matrix D such that

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix} = PDP^{-1} .$$

Solutions to Quizzes

Solution Q1 S1: We code the problem as $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$.

Adding multiples of 1st row, $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -2 & -3 & 1 & 0 \\ 0 & 0 & -2 & -3 & 0 & 1 \end{array}\right]$.

Multiplying rows by factors, $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3/2 & -1/2 & 0 \\ 0 & 0 & 1 & -3/2 & 0 & -1/2 \end{array}\right]$.

Adding multiples of 3rd row, $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 3/2 & 0 & -1/2 \end{array}\right]$.

Adding multiples of 2nd, $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 3/2 & 0 & -1/2 \end{array}\right]$.

So the inverse is $\frac{1}{2}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix}$.

Solution Q1 S2: The problem codes as $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$.

Subtracting the 1st row from the 3rd, $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$.

Subtracting the 2nd from the 3rd, $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$.

Subtracting the 3rd from the others, $\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$.

Subtracting the 2nd from the 1st, $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$.

So the answer is $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

Solution Q2 S1: The distance is $\dfrac{|2.3 + 4.4 - 3.4 - 6|}{\sqrt{2^2 + 4^2 + 3^2}} = 4/\sqrt{29}$.

Solution Q2 S2: The distance is $\dfrac{|1.5 + 1.5 + 3.2 - 10|}{\sqrt{1^2 + 1^2 + 3^2}} = 6/\sqrt{11}$.

Solution Q3 S1: Applying row operations to the matrix

$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$, we obtain first $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$, then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Solving, we have $y = -2z$ and $x = -y - z = z$. Thus, $(x, y, z) = z(1, -2, 1)$. So the solution space has basis $\{(1, -2, 1)\}$ and dimension 1.

Solution Q3 S2: Applying row operations to $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, we obtain $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$, then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Solving as in Q3 S1, we obtain $(x, y, z) = z(1, -2, 1)$. As before, the solution space has basis $\{(1, -2, 1)\}$ and dimension 1.

Solution Q4 S1: A basis for S is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Evidently, $\dim(S) = 6$.

Solution Q4 S2: A basis for S is $\{f_0, f_2, f_4, f_6\}$ where $f_m(x) = x^m$. In particular, $\dim(S) = 4$.

Solution Q5 S1: The characteristic equation is

$$0 = (3 - \lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) .$$

The solutions are $\lambda \in \{2, 4\}$.

Solution Q5 S2: The characteristic equation is

$$0 = (1 - \lambda)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2) .$$

The solutions are $\lambda \in \{0, 2\}$.

Solution Q6 S1: By inspection (or as in Quiz 5 Section 2), the eigenvalues are 0 and 2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So we can put $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution Q6 S2: By inspection, the eigenvalues are 1, 2, 3. The corresponding eigenvectors are $\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, $P = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.