

Below is a record of the exam questions, together with solutions and comments. The duration of the exam was two hours. All the questions had equal weight.

**1:** Show that, if  $\{\vec{u}, \vec{v}, \vec{w}\}$  is a basis for a finite dimensional vector space  $V$ , then  $\{\vec{u} - 2\vec{v} + 3\vec{w}, 2\vec{u} + \vec{v} - \vec{w}, \vec{u} - \vec{v} + \vec{w}\}$  is also a basis for  $V$ .

**2:** Let  $V$  be the vector space of all functions from  $\mathbb{R}$  to  $\mathbb{R}$  with usual definitions of addition and scalar multiplication:

$$(f \oplus g)(x) = f(x) + g(x), \quad (c \odot f)(x) = cf(x), \quad \text{where } c \text{ is a scalar.}$$

Show that,

**a)** if  $W_1$  is the set of all **even** functions (i.e.  $f(x) = f(-x)$  for all  $x \in \mathbb{R}$ ) in  $V$ ,

**b)** if  $W_2$  is the set of all **odd** functions (i.e.  $f(x) = -f(-x)$  for all  $x \in \mathbb{R}$ ) in  $V$ , both  $W_1$  and  $W_2$  are subspace of  $V$ .

**3:** Find the dimension of the real vector space

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ x \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 8 \\ 2x \end{bmatrix} \right\}$$

as a subset of  $\mathbb{R}^4$ , where  $x$  is a real number. (Hint: The answer depends on  $x$ .)

**4:** Let  $\{\vec{e}_1, \dots, \vec{e}_m\}$  and  $\{\vec{f}_1, \dots, \vec{f}_n\}$  be linearly independent subsets of a vector space  $V$ , and suppose that  $\text{span}\{\vec{e}_1, \dots, \vec{e}_m\} \cap \text{span}\{\vec{f}_1, \dots, \vec{f}_n\} = \{\vec{0}\}$ . Show that the set  $\{\vec{e}_1, \dots, \vec{e}_m, \vec{f}_1, \dots, \vec{f}_n\}$  is linearly independent.

**5:** Let  $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$ . Find the rank and nullity of  $\mathbf{A}$ .

### Solutions and comments:

**1:** Let  $T = \{\vec{u} - 2\vec{v} + 3\vec{w}, 2\vec{u} + \vec{v} - \vec{w}, \vec{u} - \vec{v} + \vec{w}\}$ . We must show that any vector  $\vec{t} \in V$  can be written uniquely as a linear combination of the elements of  $T$ . The set  $S = \{\vec{u}, \vec{v}, \vec{w}\}$  is a basis for  $V$ , so there exist unique real numbers  $b_1, b_2, b_3$  such that

$$\vec{t} = b_1\vec{u} + b_2\vec{v} + b_3\vec{w}.$$

But the matrix  $\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$  is non-singular, because its determinant is

$$1(1 - 1) - 2(-2 + 3) + 1(2 - 3) = -3 \neq 0.$$

In other words, the system of equations

$$a_1 + 2a_2 + a_3 = b_1, \quad -2a_1 + a_2 - a_3 = b_2, \quad 3a_1 - a_2 + a_3 = b_3$$

has a unique solution in  $a_1, a_2, a_3$  for any given  $b_1, b_2, b_3$ . Therefore, as required,  $\vec{t}$  can be written uniquely in the form

$$\begin{aligned} \vec{t} &= (a_1 + 2a_2 + a_3)\vec{u} + (-2a_1 + a_2 - a_3)\vec{v} + (3a_1 - a_2 + a_3)\vec{w} \\ &= a_1(\vec{u} - 2\vec{v} + 3\vec{w}) + a_2(2\vec{u} + \vec{v} - \vec{w}) + a_3(\vec{u} - \vec{v} + \vec{w}). \end{aligned}$$

Alternatively, one can show separately that  $T$  spans  $V$  and that  $T$  is linearly independent.

Or as another variation, one can observe that, since  $|T| = 3 = \dim(V)$ , the spanning property of  $T$  is equivalent to the linear independence property of  $T$ . Hence, it suffices to show only one of those two properties.

**2:** Let  $f$  and  $g$  be vectors in  $W_2$ , and let  $c$  be a scalar. To show that  $W_2$  is a subspace of  $V$ , we must check that  $f \oplus g$  and  $c \odot f$  belong to  $W_2$ . We have

$$\begin{aligned} (f \oplus g)(-x) &= f(-x) + g(-x) = -f(x) - g(x) = -(f(x) \oplus g(x)) = -(f \oplus g)(x), \\ (c \odot f)(-x) &= cf(-x) = -cf(x) = -(c \odot f)(x). \end{aligned}$$

Hence  $f \oplus g \in W_2$  and  $c \odot f \in W_2$ . This completes the proof that  $W_2$  is a subspace of  $V$ . The proof of the conclusion for  $W_1$  is similar.

**3:** By routine methods: We shall show that the dimension of the span is 3 when  $x = 5$  and it is 2 otherwise. The dimension is the rank of the matrix

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 6 \\ 1 & 3 & 4 & 8 \\ 1 & 4 & x & 2x \end{bmatrix}.$$

Elementary row operations do not change the rank. Applying elementary row operations, we can replace the matrix with

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & x - 5 & 2x - 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Evidently the rank is as asserted above.

By direct argument: Let  $v_1, v_2, v_3, v_4$  be the four specified vectors, in order. Since  $v_4 = 2v_3$ , we have  $\text{span}\{v_1, v_2, v_3, v_4\} = \text{span}\{v_1, v_2, v_3\}$ . Plainly, the set  $\{v_1, v_2\}$  is linearly independent. The equality

$$\lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

has a unique solution, namely  $\lambda_1 = \lambda_2 = 1$ . So, if  $x = 5$ , then the equality  $\lambda_1 v_1 + \lambda_2 v_2 = v_3$  has a solution, namely  $\lambda_1 = \lambda_2 = 1$ , hence  $\dim \text{span}\{v_1, \dots, v_4\} = 2$ . On the other hand, if  $x \neq 5$  then the equality has no solution in  $\lambda_1$  and  $\lambda_2$ , hence  $\dim \text{span}\{v_1, \dots, v_4\} = 3$ .

**4:** Let  $a_1, \dots, a_m$  and  $b_1, \dots, b_n$  be real numbers and suppose that  $\sum_i a_i e_i + \sum_j b_j f_j = 0$ . We are to show that each  $a_i = 0$  and each  $b_j = 0$ . Now

$$\sum_i a_i e_i = - \sum_j b_j f_j \in \text{span}\{e_1, \dots, e_m\} \cap \text{span}\{f_1, \dots, f_n\} = \{0\}.$$

So  $\sum_i a_i e_i = 0$  and  $\sum_j b_j f_j = 0$ . But  $\{e_1, \dots, e_n\}$  is linearly independent, so each  $a_i = 0$ . Similarly, each  $b_j = 0$ .

Comments: Some common mistakes are listed below.

**4.A:** Many candidates wrote down suitable equations, but with absent or incorrect indications as to the logical relationships between the equations, for instance,

$$\text{“}\sum_{i=1}^n a_i e_i = 0 \text{ when } a_1 = a_2 = \dots = a_n = 0\text{”}.$$

Some candidates just wrote down loads of equations connected by mysterious arrows. As has been stressed in class, that does not constitute a deductive argument, To convey a mathematical argument clearly and unambiguously, one should use complete, grammatically correct sentences.

**4.B:** The crux of the argument is to explain why the equality  $\sum_i a_i e_i + \sum_j b_j f_j = 0$  implies the equalities  $\sum_i a_i e_i = 0$  and  $\sum_j b_j f_j = 0$ . Very many candidates gave no explanation at all. Many candidates failed to adequately explain how they made use of the hypothesis  $\text{span}\{e_1, \dots\} \cap \text{span}\{f_1, \dots\} = \{0\}$ . One candidate wrote along the lines:

“None of the  $e_i$  and no combination of the  $e_i$  is an  $f_i$  or a combination of the  $f_i$ .”

That does just about succeed in conveying the idea, although it is clumsy and not quite correct: the zero vector is a linear combination of the  $e_i$  and it is also a linear combination of the  $f_j$ . However, two candidates expressed variants of the assertion:

“None of the  $e_i$  is in the span of the  $f_j$  and none of the  $f_j$  is in the span of the  $e_i$ .”

That weakening of the hypothesis is insufficient. A counter-example is the case

$$\begin{aligned} e_1 &= (1, 0, 0, 0, 0), & e_2 &= (0, 1, 0, 0, 0), \\ f_1 &= (1, 1, 1, -1, 0), & f_2 &= (1, 1, 0, 1, -1), & f_3 &= (1, 1, -1, 0, 1). \end{aligned}$$

Here,  $\{e_1, e_2\}$  and  $\{f_1, f_2, f_3\}$  are linearly independent and the condition in the latest quote is satisfied, but  $\{e_1, e_2, f_1, f_2, f_3\}$  is not linearly independent since  $3e_1 + 3e_2 - f_1 - f_2 - f_3 = 0$ .

**4C:** A few candidates argued that  $\{e_1, \dots, e_m, f_1, \dots, f_i\}$  and  $\{f_{i+1}, \dots, f_n\}$  are linearly independent for all  $i$ . But, to do that successfully, either one must include the condition

$$\text{span}\{e_1, \dots, e_n, f_1, \dots, f_i\} \cap \text{span}\{f_{i+1}, \dots, f_m\} = \{0\}$$

as part of the inductive assumption, or else one must make use of the condition  $\text{span}\{e_1, \dots\} \cap \text{span}\{f_1, \dots\} = \{0\}$  in some other way. But neither of those two approaches escapes the need to deal with the crux of the problem. Thus, this inductive approach is a *red herring*, and it does not make the problem any easier.

**5:** Plainly, any three of the four columns are linearly independent as vectors in  $\mathbb{R}^4$ , so the rank is at least 3. But the sum of the columns is the zero vector, so the rank is exactly 3. Therefore the nullity is  $4 - 3 = 1$ .

As an alternative solution, it is easy to see that a vector  $(x_1, x_2, x_3, x_4)$  belongs to the null space if and only if  $x_1 = x_2 = x_3 = x_4$ . So the nullity is 1. It follows that the rank is  $4 - 1 = 3$ .

The question can also be done in a routine way by using elementary row operations to reduce to a matrix in echelon form.