Past Papers for “Discrete” Courses

Bilkent University, Laurence Barker, 19 January 2016

In a few places, I have made corrections or improvements to the questions.
Other “discrete” past papers can be found in files named “arch110...” on my homepage.
The past exam papers recorded below are relevant to the Bilkent courses:

MATH 110 Discrete Mathematics,
MATH 132 Discrete and Combinatorial Mathematics,
MATH 210 Finite and Discrete Mathematics.

Those three courses have different emphases on techniques, but their syllabi have a considerable overlap of topics.

Discrete mathematics tends to be very different from calculus, statistics and linear algebra. In discrete mathematics, very often, no systematic procedure can be applied, so one has to put a lot of effort into clear explanation.

For all three courses, the main aim is to acquire the skill of very clear deductive explanation. A very clear deductive explanation is called a *proof*. For an argument to be a *proof*, the basic criterion is that most of the members of the class would be able to follow it easily if they were to read it carefully.

Proofs come in the form of text, possibly with diagrams. The text is a sequence of complete sentences. Your intelligent but inimaginative reader must be able to work through it, step-by-step, sentence-by-sentence, stopping at the end of each sentence to make sure that the argument thus far makes sense and is correct. If your attempted proof looks like a plate of spaghetti and if, on closer inspection, it turns out to be a collage of equations connected by arrows, then you are doing it wrong because your reader will not know where to start.

Many of the homeworks are taken from:

R. P. Grimaldi, “Discrete and Combinatorial
Mathematics”, 5th Edition, (Pearson, 2004).

I strongly recommend that textbook. It has a huge collection of good exercises and, at the back, helpful sketch-notes of solutions to the odd-numbered exercises. *Do be aware, however, that those sketch-notes are not complete solutions! A complete response to a mathematical exercise always includes a very clear deductive explanation, except when the answer is obvious.*

There is no such thing as a “model solution”. The solutions supplied below are just *examples* of responses to the exercises and exam questions. Often, a mathematical assertion has many different good proofs. Besides, different mathematicians have different styles.

You will not get far by imitation. Sure enough, imitation does confer some short-term benefits. In introductory mathematics, some weak students do sometimes manage to scrape passes by imitating without much understanding. But that approach leads to a dead-end. In the long run, when learning the art of proof, it is best to explain material in the way that *you* think is most clear, even if that occasionally results in stupid teachers seeming to subtract marks unfairly. In mathematics, *you* are the ultimate arbiter of proof.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Midterm, Fall 2015, Bilkent, LJB, 18 November 2015

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum five sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

Question 1: 10% The game of simplified nim begins with two piles of matches. Each pile has the same number of matches. Player A moves first, then Player B, then A, then B, and so on. On each move, the player must remove some or all of the matches from one of the piles, and may not remove any matches from the other pile. The winner is the player who removes the last match. Using argument by induction, show that, if Player B plays correctly, then Player B will win.

Question 2: 20% Let x_0, x_1, ..., be an infinite sequence such that $x_0 = 1$ and $x_1 = -2$ and $x_{n+2} + 2x_{n+1} + x_n = 0$ for all $n \geq 0$. Find a formula for x_n.

Question 3: 30% Let G_1 and G_2 be conected graphs with no shared vertices. Let x_1 and x_2 be vertices of G_1 and G_2 respectively. Let G be the graph consisting of the vertices and edges of G_1 and G_2 together with the edge x_1x_2.
(a) Is it true that, if G_1 and G_2 both have Euler circuits, then G must have an Euler path but no Euler circuit? (Give a proof or a counter-example.)
(b) Is it true that, if G has an Euler path but no Euler circuit, then G_1 and G_2 must both have Euler circuits? (Give a proof or a counter-example.)
(c)[1] Suppose that G_1 and G_2 both have 7 vertices and 21 edges. Find an Euler path for G, describing your Euler path by listing the vertices in order. (Diagrams with tiny handwriting will not be marked.)

Question 4: 20% Let m and n be positive integers with $m \geq n$. How many ways are there of putting m plain balls into n coloured boxes such that no box is left empty? (We understand plain to mean indistinguishable and coloured to mean distinguishable.)

Question 5: 20% Let G be a connected planar graph with n vertices and e edges.
(a) State, without proof, a relationship between n and e. Use it to deduce that some vertex of G has degree less than 6.
(b) Show that, for any planar diagram of G, some face shares an edge with less than 6 other faces.

[1] In the original exam paper, this question had a typo.

Solutions to MATH 132 Midterm Fall 2015

1: We must prove that, for all positive integers n, the position with n matches in both piles is a losing position. We shall argue by induction on n. The case $n = 1$ is trivial. Now suppose that $n \geq 2$ and that the assertion holds for all positive integers less than n. Letting m be the number of matches removed by the first player, then then other player should remove m matches from the other pile, thus obtaining the position with $n - m$ matches in both piles, which is a losing position by the inductive assumption.

Comment: This is an argument by so-called "strong induction". The inductive assumption is "$n \geq 2$ and ... the assertion holds for all positive integers less than n". Note that the argument does not work with the "weak induction" assumption "$n \geq 2$ and the assertion holds with n replaced by $n - 1$". If you failed to state any inductive assumption, then your argument is incomplete.

There were 5 marks for finding the winning strategy.

Some candidates did not understand the meaning of the term "inductive assumption". The term is the name of the assumption at the beginning of the "inductive step". When doing the "inductive step", you start with the inductive assumption, then you make deductions from that until you arrive at the next term of the sequence of statements that you are trying to prove.

Below, after this section on solutions and coments, four outstanding induction arguments are quoted from exam scripts.

2: The quadratic equation $X^2 + 2X + 1 = 0$ has unique solution $X = -1$. Therefore $x_n = (C + nD)(-1)^n$ for some C and D. We have $1 = x_0 = C$ and $-2 = (C + D)(-1)^1$, hence $D = 1$ and $x_n = (1 + n)(-1)^n$.

Comment: This is a routine question. Recall, to obtain grade C, you should be able to do this kind of question, and you should also be able to partially deal with some more difficult problem questions and some easy theoretical questions.

Almost all the marks are for getting the method right and presenting it clearly. For mistakenly finding repeated solution $X = 1$, then making another mistake of calculation to arrive at $C = 1 = -D$, concluding that $x_n = 1 - n$, the mark would be 16 out of 20.

3: Part (a). Yes. Suppose that G_1 and G_2 have Euler circuits C_1 and C_2. We may assume that C_1 starts and finishes at x_1 and that C_2 starts and finishes at x_2. Then the G has an Euler path consisting of C_1 followed by the edge $x_1 x_2$ followed by C_2.

Part (b). No. One counter-example is the case where G_1 has 2 vertices and G_2 has 1 vertex.

Part (c)[2]. Let a_0, ..., a_6 be the vertices of G_1. Let b_0, ..., b_6 be the vertices of G_2. Since G_1 has $7.6/2 = 21$ edges, G_1 is a copy of K_7. Renumbering if necessary, we may assume that $x_1 = a_0$. Similarly, G_2 is a copy of K_7 and we may assume that $x_2 = b_0$. One Euler path of G is

$$a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_0, a_2, a_4, a_6, a_1, a_3, a_5, a_0, a_3, a_6, a_2, a_5, a_1, a_4, a_0,$$

$$b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_0, b_2, b_4, b_6, b_1, b_3, b_5, b_0, b_3, b_6, b_2, b_5, b_1, b_4, b_0.$$

[2]Because of the typo in the original exam paper, no marks were awarded for this part.

Alternative for part (a): If G_1 and G_2 have Euler circuits, then all their vertices have even degree, hence x_1 and y_1 are the only two vertices of G with odd degree. Since G is connected, it follows that G has an Euler path from x_1 to x_2.

Comment: Many candidates lost 1 mark for failing to state the answer to (a), another mark for failing to state the answer to (b). The answer to (a) is "Yes" or "It is true". The answer to (b) is "No" or "It is false". You should state answers or conclusions at the beginning or at the end of your explanation. Your reader does not wish to expend energy trying to deduce your answer from a long paragraph of discussion.

To prove a universal positive assertion, say, "All ravens are black", you must present a general proof that, given any raven, then it is black. To refute a universal positive assertion, you need only present a counter-example, a non-black raven.

Thus, part (a) requires a general proof about any two graphs G_1 and G_2 that have Euler circuits. But part (b) just requires you to specify one counter-example. In part (b), it is not helpful to give a story about why some particular line of reasoning would not work. The discovery of a flaw in an argument does not imply that the conclusion of the argument is false. Can you see the huge logical flaw in the following argument? "Ravens haunt the darkness of the night. However, this does not imply that all ravens are black. Therefore, some raven is non-black."

4: By placing one ball in each box and then considering the possible arrangements of the other balls, we see that the answer is the number of ways of placing $m - n$ balls in n boxes. By a standard formula, this number is $\binom{(m-n)+(n-1)}{n-1} = \binom{m-1}{n-1}$.

Comment: Of course, the answer can also be expressed as $\binom{m-1}{m-n}$.

Many candidates gave, as answer, the memorized formula $\binom{n+m-1}{n-1}$. But this is the standard answer to the similar question where there is no requirement that each box has at least one ball. One cannot do this kind of question if one remembers only the standard formula. It is better to forget the formula and to understand the idea behind it.

5: Part (a). Recall, $e \leq 1$ or $e \leq 3n - 6$. Let x_1, ..., x_n be the vertices. Trivially, if $e \leq 1$ then $d(x_1) < 6$. Otherwise, $d(x_1) + ... + d(x_n) = 2e \leq 6n - 12$. Since the average of the degrees $d(x_i)$ is less than 6, at least one of the $d(x_i)$ is less than 6.

Part (b). Let $\widehat{G}$ be a planar diagram representing G. Let F be the graph such that the vertices of F are the faces of $\widehat{G}$, two vertices of F being adjacent provided, as faces of $\widehat{G}$, they share at least one edge. By regarding G as a map of countries, reinterpreting the vertices of F as capital cities and reinterpreting the edges of F as border-crossing roads between capital cities, we see that F is connected and planar. The required conclusion now follows by applying part (a) to F.

Comment: Of 139 registered students, only 2 were able to do part (b). We saw the key idea in lectures, when we were discussing duality of Platonic solids.

Four responses to Question 1: how not to apply mathematical induction

The four candidates whose solutions appear below need not care about it. The best mathematicians can make awful howlers, especially when under pressure to think quickly. I presume that, instead of writing nothing, these four candidates were trying to pick up a mark or two by random luck. Nevertheless, imitation is no substitute for understanding.

Conceptual style:

Proof by induction. Base case: Player B plays correctly and Player A plays poorly and Player B wins. Inductive step: we assume that Player B plays correctly on every move and wins the game and Player A plays poorly, and this assumption is our inductive hypothesis. □

Conceptual style:

First move m_1 is played correctly by player B. Let's assume any move m_k for all k is played correctly by player B. If player B plays move m_{k+1} correctly, player B wins the game. For each correct move if the next move is correct all moves are correct. □

Analytically formal style:

$$a_n = \underbrace{1+2+3+...+k}_{\text{matches}} = k(k+1)/2$$

$$S_i = 1+2+3+...+n = n(n+1)/2$$

$$P(k+1) = (k+1)(k+2)/2$$

1) $1 \stackrel{?}{=} 1$ ✓

2) $1+2+3+...+k+(k+1) = \dfrac{k(k+1)}{2} + (k+1)$

$$= k^2 + k + 2k + 2 = \frac{k^2+3k+2}{2} = \frac{(k+2)(k+1)}{2} \checkmark \quad \square$$

Logically formal style:

$A \to a$ times n (piles) $\longrightarrow S_n$

$B \to a$ times $n-1$ (piles) $\longrightarrow$ let's say T_n

$$S_n - S_{n-1} = T_n - T_{n-1}$$

$$na - (na - a) = an - a - (an - 2a)$$

□

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Midterm Makeup, Fall 2015, Bilkent, LJB, 21 December 2015

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum five sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

Question 1: 20% Recall that the Fibonacci numbers F_0, F_1, ... are defined by the condition that $F_0 = 0$ and $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Show that F_{4m} is divisible by 3 for all positive integers m.

Question 2: 20% Let x_0, x_1, ... be an infinite sequence such that $x_0 = x_1 = 1$ and $x_{n+2} - 2x_{n+1} + 2x_n = 0$ for all $n \geq 0$. Consider an integer $m \geq 0$. Give a formula for x_{4m} that does not involve complex numbers.

Question 3: 20% We have 8 red balls, which are indistinguishable from each other. We also have 15 blue balls, which are indistinguishable from each other. How many ways are there of putting the balls into 4 numbered boxes such that, for each box, the number of blue balls is greater than the number of red balls?

Question 4: 20% A **cycle**, recall, is a circuit that has no repeated vertices except that the last vertex is the same as the first vertex. Let G be a graph. Show that every vertex of G has even degree if and only if the edges of G can be coloured in such a way that, for each colour, the edges with that colour form a cycle.

Question 5: 20% Let G be a planar graph with at least 3 edges. Let n and e be the number of vertices and the number of edges, respectively. Show that $e = 3n - 6$ if and only if it is impossible to add a new edge such that the new graph is planar.

Solutions to MATH 132 Midterm Makeup, Fall 2015

1: We shall prove, by induction, that $F_{4m} \equiv 0$ and $F_{4m+1} \equiv 1$ modulo 3. The case $m = 0$ is given by the initial conditions. Now suppose that $m \geq 1$ and that $F_{4m-4} \equiv 0$ and $F_{4m-3} \equiv 1$. Then $F_{4m-2} \equiv 1$, hence $F_{4m-1} \equiv 2$, yielding the required conclusion. $\square$

2: The quadratic equation $X^2 - 2X + 2 = 0$ has solutions $1 + i$ and $1 - i$. So $x_n = A(1 + i)^n + B(1 - i)^n$ for some A and B. The initial conditions yield $A = B = 1/2$. Therefore

$$x_{4m} = ((1 + i)^{4m} + (1 - i)^{4m})/2 = (-4)^m \ .$$

3: Put one blue ball in each box, then glue each red ball to one of the remaining blue balls. The number of ways of putting the 8 glued pairs of balls into the boxes times the number of ways of putting the last 3 blue balls into the boxes is

$$\binom{8+4-1}{8}\binom{3+4-1}{3} = \binom{11}{8}\binom{6}{3}$$

$$= \frac{11\,.\,10\,.\,9\,.\,6\,.\,5\,.\,4}{3\,.\,2\,.\,3\,.\,2} = 11\,.\,5\,.\,3\,.\,5\,.\,4 = 11\,.\,3\,.\,100 = 3300\ .$$

4: Suppose that every vertex of G has even degree. Arguing by induction on the number of edges e, we shall prove that the edges of G can be coloured as specified. The case $e = 0$ is trivial. Now suppose that $e \neq 0$ and that the assertion holds in all cases with fewer edges. Any forest with at least one edge has at least one vertex with degree 1. So G is not a forest, in other words, G has a cycle of positive length. Colour the edges of that cycle red. Removing the red edges, we obtain a graph H satisfying the required conditions and with fewer edges. By the inductive assumption, we can colour the edges of H in the required way, without using red. Half of the required conclusion is now established. The converse half is obvious. $\square$

5: Suppose it is impossible to add a further edge. Then G must be connected, every face must be a triangle, and every edge must have two distinct faces on either side. The number of pairs (ϵ, F), where ϵ is an edge of face F, is $2e = 3f$. But $n - e + f = 2$, hence $e = 3n - 6$.

Conversely, suppose that $e = 3n - 6$. Let G_1, ..., G_k be the connected components of G. Let n_i and e_i, respectively, be the number of vertices and edges of G_i. If G is a forest, then $e < n$, hence $e \geq 3(e + 1) - 6 = 3e - 3$, contradicting the condition that $e \geq 3$. So G is not a forest and we may assume that G_1 is not a tree, hence $e_1 \leq 3n_1 - 6$. For $i \neq 1$, we have $e_i < n_i$ when G_i is a tree, $e_i \leq 3n_i - 6$ otherwise. The equality $e = 3n - 6$ now implies that $k = 1$, in other words, G is connected.

Let f be the number of faces of G, and let c be the average length of the circuit around the border of a face. A counting argument as above, counting (ϵ, F) twice when F is on both sides of ϵ, yields $2e = cf$. Again using $n - e + f = 2$, we obtain $n - 2 = e(1 - 2/c)$, hence $3n - 6 = e = (n - 2)c/(c - 2)$. But $c \geq 3$ and the function $c \mapsto c/(c - 2)$ is strictly monotonically decreasing, hence $c = 3$, in other words, every face is a triangle.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS

Final, Fall 2015, Bilkent, LJB

2 January 2016

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum five sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
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All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please do not hand in the question sheet! The examiner already knows the questions.

Question 1: 40% Consider the coding scheme such that the message words and received words are binary strings and the generating matrix is $G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$.
(a) Write down the 8 message words and their corresponding codewords.
(b) Without writing down the decoding table, explain why the decoding function cannot use all of the weight 1 binary strings as coset leaders. (Recall, the *coset leaders* are the received words that appear in the column of the decoding table underneath the message word with weight zero.)
(c) Making use of syndromes, and without writing down the decoding table, show that the binary strings

000000 , 000001 , 000010 , 000100 , 001000 , 010000 , 000011 , 000110

can be used as the coset leaders.
(d) Without writing down the decoding table, decode the words 000000, 111000, 000101.
(e) How many errors of transmission can be detected? How many errors of transmission can be corrected? Briefly justify your answers.

Question 2: 20% How many ways are there of putting 9 distinguishable balls into 6 indistinguishable boxes with no box left empty and then putting the boxes into 3 indistinguishable bags with no bag left empty?

Question 3: 20% Given a partial ordering $\leq$ on a set X, the *Hasse graph* of $\leq$ is the graph such that X is the set of vertices and any Hasse diagram of $\leq$ is also a diagram of the graph.
(a) Up to isomorphism, how many trees with 4 vertices are there?
(b) Up to isomorphism, how many partial orderings $\leq$ on a set of size 4 are there such that the Hasse graph of $\leq$ is a tree?

Question 4: 20% Let X be a set and let f be a function $X \to X$. For integers $n \geq 0$, we define f^n to be the function $X \to X$ such that $f^0 = \mathrm{id}_X$ (the identity function on X) and $f^{n+1} = f \circ f^n$ (thus $f^1(x) = f(x)$ and $f_2(x) = f(f(x))$ and $f^3(x) = f(f(f(x)))$ and so on). Suppose that, for all $x \in X$, there exists an integer $n_x \geq 1$ such that $f^{n_x}(x) = x$.
(a) Give an example such that $f^n \neq \mathrm{id}_X$ for all integers $n \geq 1$.
(b) Show that if $|X| = 2016$, then $f^n = \mathrm{id}_X$ for some integer $n \geq 1$.

Solutions to MATH 132 Final, Fall 2015

1: Part (a). The codewords for each message word are as shown:

mess. words	000	001	010	011	100	101	110	111
codewords	000000	001101	010111	011010	100111	101010	110000	111101

Part (b). The parity-check matrix is $H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$.

The weight 1 words 100000 and 010000 cannot both be used as coset leaders because they have the same syndrome, 111.

Part (c). The syndromes of the given received words are as shown.

rec. words	000000	000001	000010	000100	001000	010000	000011	000110
syndromes	000	001	010	100	101	111	011	110

By part (b), the 8 proposed coset leaders satisfy the minimum weight condition. Moreover, the 8 corresponding syndromes are mutually distinct, as required.

Part (d). The decodings are 000 and 110 and 001, respectively. The calculations are shown in the next table.

r	Hr^T	s	c	w
000000	000	000000	000000	000
111000	101	001000	110000	110
000101	101	001000	001101	001

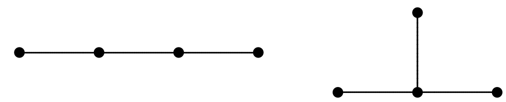
Part (e). The minimum distance between codewords is 2, so one error can be detected and no errors can be corrected.

2: There are $S(9,6)$ ways of putting the balls in the boxes. That having been done, the boxes are now distinguishable by means of the balls they contain. So there are $S(6,3)$ ways of putting the boxes in the bags.

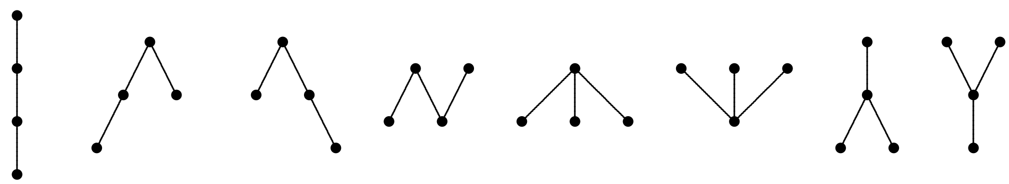
Using the recurrence relation $S(m+1,n) = S(m,n-1) + nS(m,n)$ with initial conditions $S(1,m) = S(m,m) = 1$, we obtain the following table of Stirling numbers of the second kind. (Unnecessary entries have been omitted from the table.) The number of arrangements of balls in boxes in bags is $S(9,6)S(6,3) = 2646 \,.\, 90 = 264600 - 26460 = 238140$.

$S(m,n)$		n 1	2	3	4	5	6
m	1	1					
	2	1	1				
	3	1	3	1			
	4	1	7	6	1		
	5		15	25	10	1	
	6			90	65	15	1
	7				350	140	21
	8					1050	266
	9						2646

Comment: For full marks, it is necessary to explain why the answer is $S(9,6)S(6,3)$ and then to calculate it. In the present context (an exam where use of electronic devices is not allowed), the alternating sum formula for $S(m,n)$ is not very practical. Of course, in "real life" applications too, the appropriate choice of method of calculation may depend on the available computing power.

3: Part (a). There are 2 such trees, as depicted. 

Part (b). Up to isomorphism, there are 4 partial orderings whose Hasse diagram describes a tree of the first kind. There are 4 partial orderings for the tree of the other kind. So there are 8 partial orderings satisfying the conditions specified in the question. They have Hasse diagrams as shown.



4: Part (a). Let X be the set of positive integers $\{1, 2, 3, ...\}$. We define f such that, given $x \in X$, and writing $q = r(r+1)/2 = 1 + 2 + ... + r$ where r is the smallest positive integer satisfying $x \leq q$, then

$$f(x) = \begin{cases} q - r + 1 & \text{if } x = q, \\ x + 1 & \text{otherwise.} \end{cases}$$

We have $f^r(x) = x$. In fact, for a natural number m, we have $f^m(x) = x$ if and only if m is divisible by r. Given any integer $n \geq 1$ then, choosing any r such that $r > n$, then putting $x = q$, we have $f^n(x) = q - r + n < x$, hence $f^n(x) \neq \mathrm{id}_X$.

Part (b). We shall show, more generally, that the conclusion holds whenever the set X is finite. It is easy to see that each function f^n is bijective. Since X is finite, there are only finitely many bijections $X \to X$. Therefore, $f^a = f^b$ for some integers $a > b \geq 0$. Putting $n = a - b$, then $f^n = (f^b)^{-1} \circ f^a = (f^a)^{-1} \circ f^a = \mathrm{id}_X$.

Comment: The problem is equivalent to the following: At a dinner party, each guest is seated in a chair and each chair seats a guest. Let f be a permutation of the chairs. Whenever the band finishes a song, the guests move in such a way that the guest in chair C moves to chair $f(C)$. Is it possible that every guest eventually returns to her original chair, yet the the original seating arrangement is never repeated? The above solution shows that it is possible when there are infinitely many guests, but it is not possible when there are only finitely many guests.

MATH 210: Finite and Discrete Mathematics. Midterm 1

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

Do not forget to justify your answers in terms which could be understood by people who know the background theory but are unable to do the questions themselves.

All graphs are understood to be finite ordinary graphs.

LJB, 13 March 2015, Bilkent University.

1: 10% Let G be a graph such that every vertex has degree 4 and the number of edges is 12. How many vertices does G have?

2: 10% Let G be a connected graph with at least 2 vertices. Show that there exists a vertex x of G such that, when we delete x and all its edges, the resulting graph is connected.

3: 20% The **cone** of a graph G is defined to be the graph $\Delta(G)$ that is obtained from G by adding a new vertex v and a new edge vx for each vertex x of G. Recall that the 3-cube is the graph C_3 with vertices 000, 001, 010, 011, 100, 101, 110, 111 where two vertices are adjacent provided they differ by exactly one digit. Thus, C_3 has 12 edges and its cone $\Delta(C_3)$ has 9 vertices and 20 edges. Find an Euler circuit for $\Delta(C_3)$. (Specify the Euler circuit by listing the vertices in order.)

4: 20% State, without proof, a theorem saying when a graph has an Euler circuit. Is the following statement true? "*Given any graph G, then the cone $\Delta(G)$ has an Euler circuit if and only if every vertex of G has odd degree*". Give a proof of a counter-example.

5: 20% Let G be a connected planar graph with n vertices, e edges and f faces.

(a) State a formula relating n and e and f.

(b) One proof of the correct answer to part (a) begins as follows. "*Suppose, for a contradiction, that G is a counter-example with n as small as possible. Plainly, $n \geq 2$. By Question 2, we can choose a vertex x of G such that, letting G' be the graph obtained by deleting x and all its edges from G, then G' is connected. Let n', e', f', respectively, be the number of vertices, edges and faces of G'.*" Complete this proof. (No marks will be awarded for presenting a different proof.)

6: 20% Let G be a connected planar graph with 90 edges. Suppose that, for exactly 60 of the edges, the face on one side has 3 edges and the face on the other side has 10 edges. Also suppose that, for exactly 30 of the edges, the two faces on each side are distinct from each other and both of those faces have 10 edges. How many vertices does G have?

Solutions to Midterm 1

Solution 1: Letting n be the number of vertices, then $4n = 2.12 = 24$, hence $n = 6$.

Solution 2: Remove edges from circuits until a tree is obtained. Let x be a vertex which, in the tree, has degree 1.

Solution 3: An Euler circuit is

$$v, 110, 111, 101, 100, 110, 010, 011, 001, 000, 010, v, 111, 011, v, 101, 001, v, 100, 000, v\ .$$

Comment: Specifying an Euler circuit suffices, because it is easy for the reader to check. To find an Euler circuit, the most straightforward method is to adapt the proof of the existence of Euler circuits. To find the above Euler circuit, I began with the obvious circuit

$$v, 110, 010, v, 111, 011, v, 101, 001, v, 100, 000, v$$

and then spliced in the circuits $110, 111, 101, 100, 110$ and $010, 011, 001, 000, 010$ of the two components of the remaining graph.

Solution 4: A graph Γ has an Euler circuit if and only if Γ is connected and every vertex of Γ has even degree.

The specified statement is true. If some vertex x of G has even degree, then x has odd degree in $\Delta(G)$. Hence, by the theorem, $\Delta(G)$ has no Euler circuit.

Conversely suppose that every vertex of G is of odd degree. Plainly, $\Delta(G)$ is connected and every vertex of G has even degree as a vertex of $\Delta(G)$. Finally, since the sum of the degrees of the vertices of G is even, G has an even number of vertices. So the vertex v of $\Delta(G)$ has even degree. Therefore, by the theorem, $\Delta(G)$ has an Euler circuit.

Solution 5: Part (a), $n - e + f = 2$.

Part (b). We have $n' = n - 1$ and $e' = e - d(x)$ and $f' = f - d(x) + 1$. Since $n' < n$, the minimality of G implies that $n' - e' + f' = 2$. But

$$n - e + f = (n' + 1) - (e' + d(x)) + (f' + d(x) - 1) = n' - e' + f' = 2\ .$$

This contradicts the assumption that G is a counter-example, as required.

Comment: If this were reformulated as a proof by induction, the ugly sting in the tail "this contradicts the assumption that G is a counter-example" would not be necessary.

Solution 6: Let n, e, f be the number of vertices, edges and faces, respectively. We have $e = 90$. Let f_3 and f_{10} be the number of faces with 3 edges and 10 edges, respectively. There are $60 = 3f_3$ pairs (ϵ, F) where ϵ is an edge on a face F such that F has 3 edges. There are $60 + 2.30 = 10f_{10}$ pairs (ϵ, F) where ϵ is an edge on a face F such that F has 10 edges. So $f_3 = 20$ and $f_{10} = 12$ and $f = 20 + 12 = 32$. Therefore, $n = e - f + 2 = 90 - 32 + 2 = 60$.

MATH 210: Finite and Discrete Mathematics. Midterm 2

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 17 April 2015, Bilkent University.

1: 15% Solve the recurrence relation $x_{n+2} - 2\sqrt{2}x_{n+1} + 2x_n = 0$ with initial conditions $x_0 = 1$ and $x_1 = 0$.

2: 15% Solve the recurrence relation $5x_{n+2} = 6x_{n+1} - 5x_n$ with initial conditions $x_0 = 10$ and $x_1 = 6$.

3: 30% A partial ordering $\leq$ on a set X is said to have height 2 provided there exist elements $x, y \in X$ such that $x < y$ but there do not exist elements $x, y, z \in X$ such that $x < y < z$. (The statement $x < y$ means $x \leq y$ and $x \neq y$.)

(a) Up to isomorphism, how many partial orderings of height 2 are there on a set with size 3?

(b) How many partial orderings of height 2 are there on a set with size 3?

(c) Up to isomorphism, how many partial orderings of height 2 are there on a set with size 4?

4: 25% (a) State (without proof) a recurrence relation and initial conditions for the Stirling numbers of the second kind.

(b) How many ways are there of putting 7 coloured (distinguishable) balls into 5 plain (indistinguishable) boxes such that no box is left empty? (Evaluate the answer explicitly.)

(c) How many ways are there of putting 7 coloured balls and 4 plain balls into 5 plain boxes such that every box contains at least one coloured ball?

5: 15% Let a, b, c, d be complex numbers with $a \neq 0$ and $d \neq 0$. Let x_0, x_1, ... be a sequence of complex numbers such that

$$ax_{n+3} + bx_{n+2} + cx_{n+1} + dx_n = 0$$

for all natural numbers n. Suppose that there exist complex numbers α, β, γ such that

$$at^3 + bt^2 + ct + d = a(t - \alpha)(t - \beta)(t - \gamma)$$

for all complex numbers t. Suppose, also, that there exist complex numbers A, B, C such that

$$x_0 = A + B + C \ , \quad x_1 = A\alpha + B\beta + C\gamma \ , \quad x_2 = A\alpha^2 + B\beta^2 + C\gamma^2 \ .$$

Prove, by induction, that $x_n = A\alpha^n + B\beta^n + C\gamma^n$ for all n. (Advice: remember to clearly tell the reader what your inductive assumption is.)

Solutions to Midterm 2

1: The quadratic equation $t^2 - 2\sqrt{2}t + 2 = 0$ has unique solution $t = \sqrt{2}$. So $x_n = (C + nD)\sqrt{2}^n$ for some C and D. We have $1 = x_0 = C$ and $0 = x_1 = (C + D)\sqrt{2}$. Hence $D = -1$. Therefore $x_n = (1 - n)2^{n/2}$.

2: We have $5x_{n+2} - 6x_{n+1} + 5x_n = 0$. The quadratic equation $5t^2 - 6t + 5 = 0$ has solutions α and β where

$$\alpha = \frac{6 + \sqrt{36 - 100}}{10} = \frac{3 + 4i}{5} , \qquad \beta = \frac{3 - 4i}{5} .$$

So $x_n = A\alpha^n + B\beta^n$ for some A and B. Putting $n = 0$ and $n = 1$ yields $10 = A + B$ and $6 = A(3 + 4i)/5 + B(3 - 4i)/5$. Therefore $A = B = 5$ and

$$x_n = 5\left(\frac{3 + 4i}{5}\right)^n + 5\left(\frac{3 - 4i}{5}\right)^n .$$

3: Part (a). The Hasse diagrams are as shown. In particular, there are 3 isomorphism classes.

[Cue three Hasse diagrams.]

Part (b). For the disconnected Hasse diagram, the number of partial orderings is 6. For the other two Hasse diagrams, the number of partial orderings is 3. So the total is $6 + 3 + 3 = 12$.

Part (c). There are 8 isomorphism classes, with the illustrated Hasse diagrams.

[Cue eight Hasse diagrams.]

4: Part (a), $S(m, 1) = S(m, m) = 1$ for all $1 \leq m$ and $S(m + 1, n) = S(m, n - 1) + nS(m, n)$ for all $1 < n \leq m$.

Part (b). Using part (a), we obtain the following table of Stirling numbers.

[Cue table of Stirling numbers.]

The number of arrangements is $S(7, 5) = 140$.

Part (c). After the coloured balls have been placed, the boxes become distinguishable. For each arrangement of the coloured balls there are $\binom{5-1+4}{4} = 8.7.6.5/4.3.2 = 70$ arrangements of the plan balls. So the total number of arrangements is $140.70 = 9800$.

5: Let $y_n = A\alpha^n + B\beta^n + C\gamma^n$. We shall show, by induction, that $y_n = x_n$ for all n. It is given that $y_n = x_n$ for all $n \leq 2$. Now suppose that $y_n = x_n$ and $y_{n+1} = x_{n+1}$ and $y_{n+2} = x_{n+2}$. Write $f(t) = at^3 + bt^2 + ct + d$. We have

$$ay_{n+3} + by_{n+2} + cy_{n+1} + dy_n = Af(\alpha) + Bf(\beta) + Cf(\gamma) = 0 = ax_{n+3} + bx_{n+2} + cx_{n+1} + dx_n .$$

Cancelling summands and then dividing by a, we deduce that $y_{n+3} = x_{n+3}$. □

Comment 5.1: The inductive assumption in the above proof is "Suppose that $y_n = x_n$ and $y_{n+1} = x_{n+1}$ and $y_{n+2} = x_{n+2}$." This is a vital part of the proof. If you set the argument up as "simple induction", say, with the inductive assumption "$n \geq 1$ and $y_{n-1} = x_{n-1}$", then you would not have been able to deduce that $y_n = x_n$. If you did not state any inductive assumption, then you argument is incomplete because you failed to inform the reader of the premise behind your deduction.

Comment 5.2: In the case where α, β, γ are mutually distinct, it can be shown that there always exist A, B, C as specified. So, in that case, the solution to the recurrence relation always has the specified form. But when α, β, γ are not mutually distinct, such A, B, C might not exist.

MATH 210: Finite and Discrete Mathematics. Makeup

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 13 May 2015, Bilkent University.

1: 20% (a) Let $y_n = n$. Show that

$$y_{n+2} - y_{n+1} - y_n = 1 - n \ .$$

(b) Let x_0, x_1, ... be an infinite sequence such that

$$x_{n+2} - x_{n+1} - x_n = 1 - n$$

and $x_0 = x_1 = 1$. Give an explicit formula for x_n in terms of n. (Hint: first find a recurrence relation for z_0, z_1, ... where $z_n = x_n - y_n$.)

2: 30% A **minimum element** of a partial ordering $\leq$ on a set X is an element $x_0 \in X$ such that $x_0 \leq x$ for all $x \in X$. Suppose that $|X| = 4$.

(a) How many partial orderings $\leq$ on X are there such that $\leq$ has a minimum element? (Hint: do part (b) first.)

(b) How many such partial orderings $\leq$ on X are there up to isomorphism?

3: 30% (a) Write out a table of Stirling numbers $S(m, n)$ for $8 \geq m \geq n$. What is the rule for working out the entries of the table?

(b) How many ways are there of putting 8 distinguishable objects into 3 red boxes and 2 blue boxes such that each box contains at least one ball? (The red boxes are indistinguishable from each other, likewise the blue boxes.)

4: 20% For positive integers r, m, n, let $S_r(m, n)$ be the number of ways of arranging m coloured balls in n plain boxes such that each box has at least r balls. Express $S_r(m+1, n+1)$ in terms of $S_r(m, n+1)$ and $S_r(m+1-r, n)$.

MATH 210: Finite and Discrete Mathematics. Final

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 25 May 2015, Bilkent University.

1: 40% Consider the coding scheme with encoding function $\mathbb{Z}_2^3 \to \mathbb{Z}_2^6$ given by generating matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} .$$

(a) Write down all 8 message words and their corresponding codewords.
(b) How many errors of transmission can be detected? How many errors of transmission can be corrected? Briefly justify your answers.
(c) Let 000011 be one of the coset leaders. Without working out the decoding table, find all the other coset leaders and explain why those binary strings must be the coset leaders.
(d) Without working out the decoding table, decode the the received words 011111, 111110, 000000.

2: 20% (a) Find the number of isomorphism classes of trees T such that T has exactly 7 vertices and exactly 3 of the vertices have degree 1.

(b) Let V be a set with $|V| = 7$. How many trees T are there such such V is the set of vertices of T and exactly 3 of the vertices of T have degree 1?

3: 15% Evaluate $\displaystyle\sum_{j=1}^{n} \frac{n!}{(n-j)!} S(m,j)$ for all $m \geq n \geq 1$.

4: 25% Let m be a positive integer.

(a) Let n be a positive integer with $n \geq m$. Let C be a linear code in $\mathbb{Z}_2^n$ such that $|C| = 2^m$. How many injective linear encoding functions $E : \mathbb{Z}_2^m \to C$ are there?

(b) Now suppose that $n = m + 1$. Let B be the set of elements of $\mathbb{Z}_2^{m+1}$ that are not in C. Let $b \in B$. Show that

$$B = \{b + c : c \in C\} .$$

(c) Show that, given b_1, ..., $b_r \in B$, then $b_1 + ... + b_r \in C$ if and only if r is even.

(d) How many linear codes C in $\mathbb{Z}_2^{m+1}$ are there such that $|C| = 2^m$ and the minimum distance between two distinct codewords is 2?

(e) How many injective linear encoding functions $E : \mathbb{Z}_2^m \to \mathbb{Z}_2^{m+1}$ are there such that the minimum distance between two distinct codewords is 2?

Final Solutions

1: Part (a). For the message words

$$000\,, \quad 001\,, \quad 010\,, \quad 011\,, \quad 100\,, \quad 101\,, \quad 110\,, \quad 111\,,$$

the corresponding codewords are, respectively,

$$000000\,, \quad 001101\,, \quad 010111\,, \quad 011010\,, \quad 100110\,, \quad 101011\,, \quad 110001\,, \quad 111100\,.$$

Part (b). Since the minimum weight of a nonzero codeword is 3, we can detect 2 errors and correct 1 error.

Part (c). The party-check matrix is

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} .$$

The binary strings

$$000000\,, \quad 000001\,, \quad 000010\,, \quad 000100\,, \quad 001000\,, \quad 010000\,, \quad 100000\,, \quad 000011\,,$$

have respective syndromes

$$000\,, \quad 001\,, \quad 010\,, \quad 100\,, \quad 101\,, \quad 111\,, \quad 110\,, \quad 011\,.$$

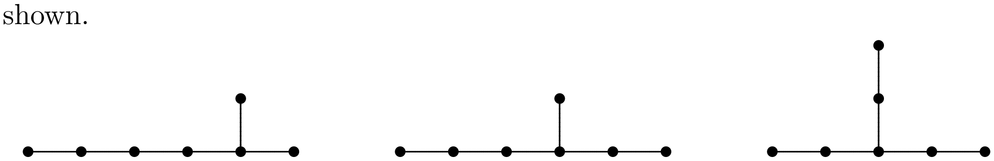
Since the received words with weight 0 or 1 have distinct syndromes, they all appear as coset leaders. Since none of those sydromes coincide with the syndrome 011 of 000011, the proposed coset leader 000011 satisfies the minimal weight condition.

Part (d). Letting r denote the received word, σ the syndrome, s the coset leader, c the codeword and w the corresponding message word, the decodings are as shown.

r	σ	s	c	w
011111	101	001000	010111	010
111110	010	000010	111100	111
000000	000	000000	000000	000

Comment: Some candidates decoded by simply noticing that each of the three given received words is at distance at most 1 from a codeword. This is valid because (as an oversight) the question did not demand that the decoding be done using the method of syndromes.

2: Part (a). For any tree with e edges and n vertices, the sum of the degress is $2e = n - 2$. So, for any tree with exactly 3 vertices of degree 1 all the other vertices have degree 2 except for one vertex with degree 3. It is now easy to see that the 3 isomorphism classes of trees T are as shown.



Part (b). The numbers of ways of assigning elements of V to the vertices of the three diagrams are, in order, $7!/2$ and $7!$ and $7!/3!$. The total is

$$7!(1/2 + 1 + 1/6) = 2520 + 5040 + 840 = 8400 \ .$$

3: The sum can be rewritten as $\sum_{j=1}^{n} \binom{n}{j} j!\, S(m,j)$. Since $\binom{n}{j}$ is the number of ways of choosing a subset $J \subseteq \{1, ..., n\}$ with size $|J| = j$ and since $j!S(m,j)$ is the number of surjections from $\{1, ..., m\}$ to J, the value of the sum is the number of functions from $\{1, ..., m\}$ to $\{1, ..., n\}$, which is n^m.

4: Part (a). Let e_i be the weight unity element of $\mathbb{Z}_2^m$ with i-th digit equal to 1. there are $2^m - 1$ choices for $E(e_1)$, then $2^m - 2$ choices for $E(e_2)$ and so on, finally $2^m - 2^{m-1}$ choices for $E(e_m)$. So the number of such E is

$$(2^m - 1)(2^m - 2)...(2^m - 2^{m-1}) = 2^{m(m-1)/2}(2^m - 1)(2^{m-1} - 1) \ ...(2 - 1) \ .$$

Part (b). Let $B' = \{b + c : c \in C\}$. We have $B' \subseteq B$ because, if $b + c$ were a codeword, then $b = (b + c) + c$ would be a codeword, contradicting the definition of b. But $|B'| = 2^m = |B|$, so $B' = B$.

Part (c). Writing $b_i = b + c_i$ where c_i is a codeword, we have $b_1 + ... + b_r = rb + c_1 + ... + c_r$. The condition follows because rb is 0 or b depending on whether r is even or odd, respectively.

Part (d). We apply part (c). Letting b_1, ..., b_r be words of weight 1 then, since they all belong to B, their sum is in C if and only if r is even. Therefore C is the set of words of even weight. In particular, there is only one such C.

Part (e). The code must be C as in part (d). So, by part (a), the answer is $(2^m - 1)(2^m - 2)...(2^m - 2^{m-1})$.

MATH 210: Finite and Discrete Mathematics. Retake Final

Time allowed: 2 hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 4 June 2015, Bilkent University.

1: 40% Let

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

be the generating matrix of a coding scheme with message words in $\mathbb{Z}_2^3$ and received words in $\mathbb{Z}_2^6$.
(a) Write down all the message words and their corresponding codewords.
(b) Write down the parity-check matrix.
(c) Find a set of coset leaders and, without working out the decoding table, explain why those binary strings can be used as the coset leaders.
(d) Using the method of syndromes, taking the coset leaders to be as in your answer to part (c), decode the the received words 010101, 101010, 111111.
(e) How many errors of transmission can be detected? How many errors of transmission can be corrected? Briefly justify your answers.

2: 20% Let X be a finite set.
(a) What do we mean when we say that two relations $\sim$ and $\approx$ on X are isomorphic? Give a precise definition.
(b) Suppose that $|X| = 7$. Find the number of isomorphism classes of equivalence relations on X.

3: 20% For $m \geq n \geq 1$, evaluate $\displaystyle\sum_{j=n}^{m} \binom{m}{j} S(j, n)$.

4: 20% Let $m \leq n$ be positive integers.
(a) What is a linear code in $\mathbb{Z}_2^n$? Give a precise definition.
(b) How many linear codes C in $\mathbb{Z}_2^n$ are there such that $|C| = 2^m$?

Solutions to Final Retake

1: Part (a). For the message words

$$000\,, \quad 001\,, \quad 010\,, \quad 011\,, \quad 100\,, \quad 101\,, \quad 110\,, \quad 111\,,$$

the corresponding codewords are, respectively,

$$000000\,, \quad 001111\,, \quad 010111\,, \quad 011000\,, \quad 100111\,, \quad 101000\,, \quad 110111\,, \quad 111111\,.$$

Part (b). The party-check matrix is

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} .$$

Part (c). We can take the coset leaders to be

$$000000\,, \quad 000001\,, \quad 000010\,, \quad 000100\,, \quad 001000\,, \quad 000110\,, \quad 000101\,, \quad 000011\,.$$

Their respective syndromes

$$000\,, \quad 001\,, \quad 010\,, \quad 100\,, \quad 111\,, \quad 110\,, \quad 101\,, \quad 011$$

are mutually distinct. The minimal weight condition is satisfied because the omitted words with weight 1 have the same syndrome as 001000.

Part (d). Letting r denote the received word, σ the syndrome, s the coset leader, c the codeword and w the corresponding message word, the decodings are as shown.

r	σ	s	c	w
010101	010	000010	010111	010
101010	010	000010	101000	101
111111	000	000000	111111	111

Part (e). Since the minimum weight of a nonzero codeword is 2, we can detect 1 error and correct no errors.

Comment: Other answers to part (c) are possible, but the answers to part (d) will still be the same.

2: Part (a). We say that $\sim$ and $\approx$ are isomorphic provided there exists a bijection f on X such that, for all $x, y \in X$, we have $x \sim y$ if and only if $f(x) \approx f(y)$.

Sketch, part (b). Two equivalence relations on X are isomorphic provided their equivalence classes have the same sizes. Any equivalence relation $\equiv$ is determined, up to isomorphism, by the finite sequence of positive integers n_1, ..., n_r where r is the number of equivalence classes of $\equiv$ and n_i is the size of the i-th largest equivalence class of equiv. For $|X| = 7$, there are 15 isomorphism classes, because that is the number of ways of expressing 7 as a sum $n_1 + ... + n^r$ of positive integers with $n_1 \geq ... \geq n_r$, indeed,

$$7 = 6+1 = 5+2 = 5+1+1 = 4+3 = 4+2+1 = 4+1+1+1 = 3+3+1$$

$$= 3+2+2 = 3+2+1+1 = 3+1+1+1+1 = 2+2+2+1 = 2+2+1+1+1$$

$$= 2 + 1 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1 \, .$$

3: The sum can be rewritten as $\sum_{j=1}^{n} \binom{n}{j} j!\, S(m,j)$. Since $\binom{n}{j}$ is the number of ways of choosing a subset $J \subseteq \{1, ..., n\}$ with size $|J| = j$ and since $j!S(m,j)$ is the number of surjections from $\{1, ..., m\}$ to J, the value of the sum is the number of functions from $\{1, ..., m\}$ to $\{1, ..., n\}$, which is n^m.

4: Part (a). A linear code in $\mathbb{Z}_2^n$ is a non-empty subset C of $\mathbb{Z}_2^n$ such that, given $c_1, c_2 \in C$, then $c_1 + c_2 \in C$.

Part (c). Given $x_1, ..., x_r \in \mathbb{Z}_2^n$, let us write $\langle x_1, ..., x_r \rangle$ to denote the smallest code in $\mathbb{Z}_2^n$ containing x_1, ..., x_r. We can choose a code C in the following way. We first choose a non-zero word b_1. Having chosen b_i for $i < m$, we choose $b_{i+1} \in \mathbb{Z}_2^n - \langle b_1, ..., b_n \rangle$. We let $C = \langle b_1, ..., b_n \rangle$.

There are $2^n - 1$ choices for b_1, then $2^n - 2$ choices for b_2, and so on, finally $2^n - 2^{m-1}$ choices for b_m. But each C appears multiple times in that way. Given C, there are $2^m - 1$ choices for b_1, then $2^m - 2$ choices for b_2, and so on, finally $2^m - 2^{m-1}$ choices for b_m. So the number of possible C is:

$$(2^n - 1)(2^n - 2)...(2^n - 2^{m-1})/(2^m - 1)(2^m - 2)...(2^m - 2^{m-1}) \, .$$

MATH 110: DISCRETE MATHEMATICS. Midterm. Fall 2014

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

Do not forget to justify your answers in terms which could be understood by people who know the background theory but are unable to do the questions themselves.

LJB, 31 October 2014, Bilkent University.

1: 20% Give formulas for the terms of the sequence x_0, x_1, x_2, ... in the following cases.

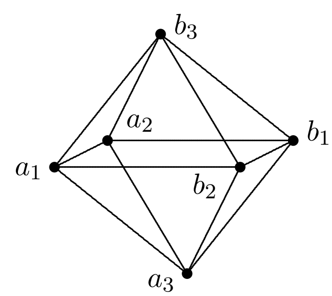
(a) $4x_{n+2} - 4x_{n+1} + x_n = 0$ with $x_0 = x_1 = 1$.

(b) $x_{n+2} + 5x_{n+1} + 6x_n = 0$ with $x_0 = 3$ and $x_1 = -8$.

(c) $x_{n+2} + (2^n + 1)x_{n+1} + (3^n + 1)x_n = 0$ with $x_0 = x_1 = 0$.

2: 20% Let a_0, a_1, a_2, ... be an infinite sequence such that $a_0 = 0$ and $a_n - a_{n-1} = n(n+1)$ for all positive integers n. Show that $a_n = n(n+1)(n+2)/3$.

3: 20% For any positive integer m, there is a graph called the m-**hyperoctahedron** with $2m$ vertices a_1, a_2, ..., a_m, b_1, b_2, ..., b_m. For each integer i in the range $1 \leq i \leq m$, the vertex a_i is adjacent to all the other vertices except b_i. For each i, the vertex b_i is adjacent to all the other vertices except a_i. The case $m = 3$ is as depicted.



(a) Find an Euler circuit for the 3-hyperoctahedron. (Specify the circuit by listing the vertices it visits in order.)

(b) Find an Euler circuit for the 4-hyperoctahedron.

(c) For which values of m does the m-hyperoctahedon have an Euler circuit?

4: 20% For which values of m is the m-hyperoctahedron a planar graph?

5: 20% **(a)** Let T be a tree such that T has exactly 6 vertices and exactly 3 of the vertices have degree 1. Using the Handshaking Lemma, find the number of vertices with degree 2 and the number of vertices with degree 3.

(b) Up to isomorphism, how many trees are there such that there are exactly 6 vertices and exactly 3 of them have degree 1?

Midterm Solutions, MATH 110, Fall 2014.

There is no such thing as a "model solution". Often, in mathematics, there are many good ways of justifying a correct conclusion.

1: Part (a). The equation $t^2 - t + 1/4$ has unique solution $t = 1/2$. So $x_n = (C + nD)/2^n$ for some C and D. We have $1 = x_0 = C$ and $1 = x_1 = (C + D)/2$. Hence $D = 1$ and $x_n = (1 + n)/2^n$.

Part (b). The equation $t^2 + 5t + 6 = 0$ has solutions $t = -2$ and $t = -3$. So $x_n = A(-2)^n + B(-3)^n$ for some A and B. Since $3 = x_0 = A + B$ and $-8 = x_1 = -2A - 3B$, we have $A = 1$ and $B = 2$. Therefore $x_n = (-2)^n + 2(-3)^n = (-1)^n(2^n + 2.3^n)$.

Part (c). Obviously, $x_n = 0$ for all n.

Comment: The answer to (c) might not be obvious at first sight but, in my view, when the answer has been stated, it becomes obvious. Some teachers might feel that, at an introductory level, an argument by induction is needed. It may be a matter of opinion. Certainly, though, at a graduate student level, the answer could just be stated and no explanation would be needed. (Note how informative the word "obviously" is in this context. It helpfully informs the reader that we have thought about whether to supply a justification. It imforms the reader that, upon deliberation, we have decided that no explanation is needed.)

2: Let $b_n = n(n + 1)(n + 2)/3$. We shall show, by induction, that $a_n = b_n$. First note that $a_0 = 0 = b_0$. Now suppose that $n \geq 1$ and that $a_{n-1} = b_{n-1}$. We have

$$b_n - b_{n-1} = (n + 2 - (n - 1))n(n + 1)/3 = n(n + 1) = a_n - a_{n-1} \; .$$

Cancelling, we deduce that $a_n = b_n$.

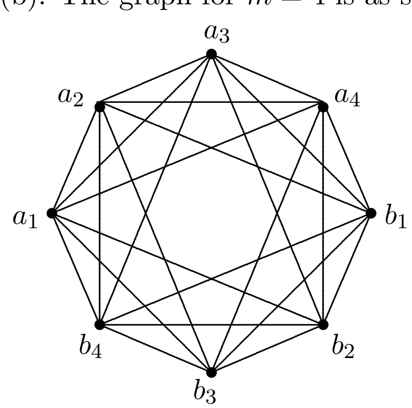
Comment: In this argument, the inductive assumption is that $n \geq 1$ and $a_{n-1} = b_{n-1}$. It is necessary to tell the reader that the conditions $n \geq 1$ and $a_{n-1} = b_{n-1}$ are being *assumed*. Otherwise, it does not make sense to *deduce* that $a_n = b_n$.

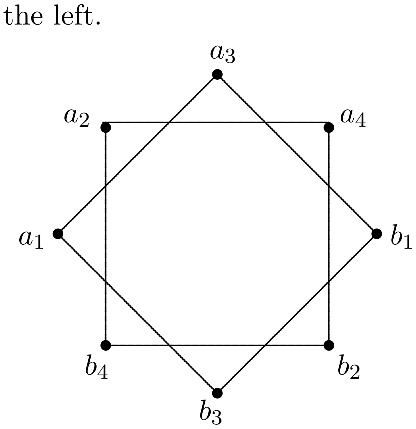
There are other ways of doing it. We could *assume* that $a_n = b_n$ and then, after showing that $b_{n+1} - b_n = a_{n+1} - a_n$, we could *deduce* that $a_{n+1} = b_{n+1}$.

As always, some people inexplicably "deduced" something without telling the reader what *assumptions* were being made. Some people mysteriously wrote $a_n - a_{n-1} = n(n + 1)$ and $a_n = n(n + 1)(n + 2)/3$ and $a_{n-1} = (n - 1)n(n + 1)/3$ near the top of the page and then, bizarrely, deduced something trivial such as $1 = 1$.

3: Part (a). One Euler circuit for $m = 3$ is a_1, b_2, b_3, a_2, a_3, b_2, b_1, a_3, a_1, b_3, b_1, a_2, a_1.

Part(b). The graph for $m = 4$ is as shown on the left.





Deleting the edges of the circuit

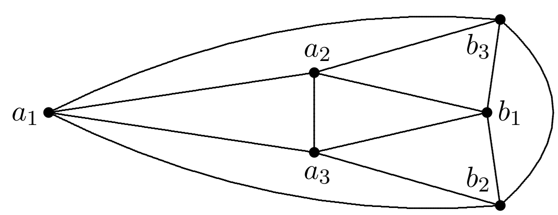
$$a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, a_1, a_4, b_3, a_2, b_1, b_4, a_3, b_2, a_1$$

we obtain the graph shown on the right, whose two components have Euler circuits a_1, a_3, b_1, b_3, a_1 and a_2, a_3, b_2, b_3, a_2. Splicing the two shorter circuits into the larger circuit, we obtain the Euler circuit

$$a_1, a_3, b_1, b_3, a_1, a_2, a_3, b_2, b_3, a_2, a_3, a_4, b_1, b_2, b_3, b_4, a_1, a_4, b_3, a_2, b_1, b_4, a_3, b_2, a_1\ .$$

Part (c). For the m-hyperoctahedron, the degree of each vertex is $2m-2$, which is always even. The graph is connected if and only if $m \geq 2$.

4: Let H_m be the m-hyperoctahedron. We shall show that H_m is planar if and only if $m \leq 3$. Since any subgraph of a planar graph is planar, and since H_n is a subgraph of H_m whenever $n \leq m$, it suffices to show that H_3 is planar and H_4 is non-planar. The next diagram shows that H_3 is planar.

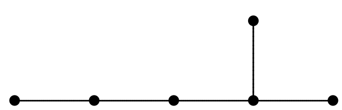


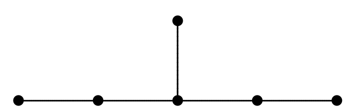
Suppose, for a contradiction, that H_4 is planar. Then $e \leq 3n-6$ where e is the number of edges of H_4 and n is the number of vertices. But $e = 8.6/2 = 24$ because $n = 8$ and all the vertices have degree 6. We deduce that $24 \leq 3.8 - 6 = 18$, a contradiction, as required.

Comment: The condition $e \leq 3n-6$ holds if and only if $m \leq 3$. This shows that the graph is non-planar when $m \geq 4$. It does not show that the graph is planar when $m \leq 3$. For $e \geq 2$, the condition $e \leq 3n-6$ is *necessary* for the graph to be planar, but it is not *sufficient*.

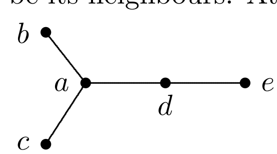
5: Part (a) Number the vertices of T from 1 to 6. Let d_i be the degree of vertex i. We can choose the numbering such that $d_1 \leq d_2 \leq ... \leq d_6$. The given condition on T is that $d_1 = d_2 = d_3 = 1$ and $d_4 \geq 2$. The number of edges of T is 5. By the Handshaking Lemma, $d_1 + d_2 + ... + d_6 = 5.2 = 10$. So $d_4 + d_5 + d_6 = 10 - 3 = 7$. Since $2 \leq d_4 \leq d_5 \leq d_6$, we must have $d_4 = d_5 = 2$ and $d + 6 = 3$. In conclusion, there are exactly 2 vertices with degree 2 and ther is exactly 1 vertex with degree 3.

Part (b). There are exactly 2 isomorphism classes of trees satisfying the specified conditions. They are as depicted.





To see this, let a be the unique vertex with degree 3 and let b, c, d be its neighbours. At least one of b, c, d must be adjacent to another vertex e. Without loss of generality, d is adjacent to e. The last vertex f must be adjacent to b or c or e. If f is adjacent to e, we obtain the first depicted tree, otherwise we obtain the second.



MATH 110: DISCRETE MATHEMATICS. Final, Fall 14/15

Time allowed: two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 3 January 2015, Bilkent University.

1: 40% Consider the encoding function with generating matrix $G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$.

(a) Write down the codewords for each of the 8 message words.
(b) Find the parity-check matrix H.
(c) Explain why there are exactly 3 binary strings with weight 2 that can appear as a coset leader. What are those 3 binary strings?
(d) Consider the decoding scheme where one of the coset leaders is the binary strong 100001. Find all the other coset leaders and write down their corresponding syndromes.
(e) For the above decoding scheme, using the method of decoding via syndromes (without writing out the decoding table), decode the received words 111111, 111000, 000111.
(f) For this code, how many single-digit errors of transmission can be detected? How many single-digit errors of transmission can be corrected?
(g) What is the rate of this code?

2: 35% **(a)** State a recurrence relation for the Stirling numbers $S(m, n)$ where m and n are positive integers.
(b) Write out a table giving the value of $S(m, n)$ for all m and n with $m \leq 8$ and $n \leq 5$.
(c) Let X be a set with $|X| = 8$. What is the number of equivalence relations $\equiv$ on X such that $\equiv$ has exactly 5 equivalence classes?
(d) What is the number of equivalence relations $\equiv$ on X such that all the equivalence classes of $\equiv$ have the same size as each other?
(e) For a function $f : X \to X$, the **image** of f, denoted $f(X)$, is the set of values of f. That is to say, $f(X) = \{f(x) : x \in X\}$. How many functions $f : X \to X$ are there such that $|f(X)| = 5$?
(f) How many functions $f : X \to X$ are there such that, for all $x \in X$, there are exactly 2 elements $y \in X$ satisfying $f(y) = f(x)$?

3: 25% Let $\leq$ be a partial ordering relation on a set S. We call $\leq$ a **local ordering** provided, for all $x, y, z \in S$ such that $x \leq y \geq z$ or $x \geq y \leq z$, we have $x \leq z$ or $z \geq x$.
(a) Let $\sim$ be the relation on S such that, given $x, y \in S$, then $x \sim y$ if and only if $x \leq y$ or $y \leq x$. Show that $\leq$ is a local ordering if and only if $\sim$ is an equivalence relation.
(b) How many isomorphism classes of local orderings are there on the set $\{1, 2, 3, 4\}$?
(c) How many local orderings are there on $\{1, 2, 3, 4\}$?

Example solutions to Final, MATH 110, Fall 2013

Reminder: Of course, there are no "model solutions" in mathematics. Mathematics thrives on diversity of methods and styles.

1: Part (a). The codewords for 000, 001, 010, 011, 100, 101, 110, 111 are, respectively,

000000, 001101, 010011, 011110, 100110, 101011, 110101, 111000 .

Part (b), $H = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$.

Part (c). The received words r of weight 0 or 1 have mutually distinct syndromes σ, as indicated in the table.

r	000000	000001	000010	000100	001000	010000	100000
σ	000	001	010	100	101	011	110

The last coset leader must have weight at least 2 and syndrome 111. There are exactly 3 ways of obtaining 111 as the sum of 2 columns of H. The 3 codewords associate with those 3 sums are 100001 and 010100 and 001010.

Part (d). Letting r run over the other 7 coset leaders, then the corresponding syndromes σ are as shown in the table in part (c).

Part (e). For $r \in \{111111, 111000, 000111\}$, the corresponding decodings w are as shown, where σ is the syndrome, s the coset leader, $c = r + s$ the codeword associated with w.

r	σ	s	c	w
111111	111	100001	0111110	011
111000	000	000000	111000	111
000111	111	100001	100110	100

Part (f). The minimum weight of a nonzero codeword is 3. So 2 errors can be detected, 1 error corrected.

Part (g). The rate is $3/6 = 1/2$.

2: Part (a), $S(m+1, n) = S(m, n-1) + nS(m, n)$ for $2 \leq n \leq n$. (The initial conditions are $S(m, 1) = S(m, m) = 1$.)

Part (b). The table is as shown.

$S(m,n)$	1	2	3	4	5	n
m 1	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	
7	1	63	301	350	140	
8	1	127	966	1701	1050	

Part (c). The number of such equivalence relations is $S(8, 5) = 1050$.

Part (d). When all the equivalence classes have size 1, the number of equivalence relations is 1. To choose an equivalence relation where every class has size 2, select an element x_0 of X, choose a partner for x_0, select an unpartnered element x_1, choose a partner for x_1, select an unpartnered element x_2, choose a partner for x_2, then partner the last two elements of X together. In that case, the number of equivalence relations is $7.5.3 = 105$. When the size is 4, the number of equivalence relations is $\binom{8}{4}/2 = 8.7.6.5/4.3.2.2 = 35$. When the size is 8, there is only 1 equivalence relation. The answer is $1 + 105 + 35 + 1 = 142$.

Part (e). To choose f, there are $\binom{8}{5} = 8.7.6/3.2 = 56$ choices for $f(X)$, then $5!S(8,5)$ choices for f. So the total number of choices for f is $56 \,.\, 5! \,.\, S(8,5) = 56.120.1050 = 7056000$.

Part (f). Let $\equiv_f$ be the equivalence relation on X such that, given $x, y \in X$, then $x \equiv y$ provided $f(x) = f(y)$. As we saw in part (d), there are 105 choices for $\equiv_f$. For each $\equiv_f$, there are $8.7.6.5 = 1680$ choices for f. So the total number of choices for f is $105.1680 = 176400$.

3: Part (a). The relation $\sim$ is reflexive and symmetric. The relation $\leq$ is a local ordering if and only if $\sim$ is transitive.

Part (b). A partial ordering is a local ordering if and only if the Hasse diagram is a disjoint union of graphs having the form $\bullet\!-\!\!-\bullet\!-\!\!-...\!-\!\!-\bullet$. Each isomorphism class corresponds to a sequence of positive integers $(x_1, ..., x_r)$ where $x_1 > ... > x_r$ and $x_1 + ... + x_r = 4$. The correspondence is such that the isomorphism class corresponds to the sequence such that the connected components of the Hasse diagram have sizes x_1, ..., x_r. Noting that

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 \,,$$

we see that the number of isomorphism classes is 5.

Part (c). When the Hasse diagram has shape 4, we mean, when the Hasse diagram is $\bullet\!-\!\!-\bullet\!-\!\!-\bullet\!-\!\!-\bullet$, the number of local orderings is $4! = 24$. When the shape is $3 + 1$, we mean, when the Hasse diagram is $\bullet\!-\!\!-\bullet\!-\!\!-\bullet \quad \bullet$, the number of local orderings is, again, 24. When the shape is $2 + 2$, there are 12 possibilities for $\leq$. When the shape is $2 + 1 + 1$, there are, again, 12 possibilities. When the shape is $1 + 1 + 1 + 1$, there is exactly 1 possibility. So the number of local orderings is $24 + 24 + 12 + 12 + 1 = 73$.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Midterm, Spring 2014, Bilkent, LJB, 2 April 2014

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum five sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

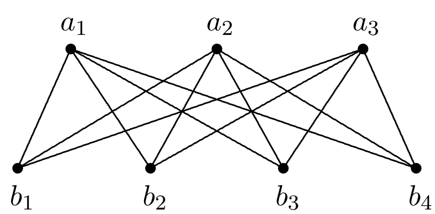
Question 1: 20% Show that, given any natural number n, then

$$1^2 + 3^2 + 5^2 + ... + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3 \ .$$

Question 2: 20% (a) Let $y_n = n$. Show that $y_{n+2} - 6y_{n+1} + 9y_n = 4(n-1)$ for all natural numbers n.
(b) Solve $z_{n+2} - 6z_{n+1} + 9z_n = 0$ with initial conditions $z_0 = 1$ and $z_1 = 6$.
(c) Solve $x_{n+2} - 6x_{n+1} + 9x_n = 4(n-1)$ with initial conditions $x_0 = 1$ and $x_1 = 7$.

Question 3: 20% For a positive integer n, the graph K_n has n vertices and $n(n-1)/2$ edges. For which values of n is it possible to colour the edges of K_n, each edge coloured either red or blue, such that the graph with the red edges has no cycles and the graph with the blue edges has no cycles? (Do not forget to justify your answer with a very clear deductive explanation.)

Question 4: 20% For integers $1 \leq m \leq n$, the graph $K_{m,n}$ has vertices a_1, a_2, ..., a_m, b_1, b_2, ..., b_n and precisely mn edges, each edge joining a vertex a_i to a vertex b_j. The graph $K_{3,4}$ is depicted.



(a) Give a complete statement of a theorem about existence of Euler paths and Euler circuits.
(b) For which positive integers m and n, with $m \leq n$, does the graph $K_{m,n}$ have an Euler path? (The answer should be obvious from part (a). No further explanation is required.)
(c) Find an Euler path for the graph $K_{4,4}$. Specify the Euler path by listing the vertices in order. (If you try to specify the path by drawing the graph and labelling the edges, then the diagram may be difficult to read and marks may be subtracted.)

Question 5: 20% Let G be a connected planar graph. Suppose that every vertex has the same degree d and that, for every face, the edges of that face form a cycle with length c. (Helpful comment: these conditions ensure that every edge has two different faces on each side.)
(a) State, without proof, a formula relating the number of vertices n, the number of edges e, the number of faces f.
(b) By considering the pairs (ϵ, F), where ϵ is an edge on face F, explain why $2e = cf$. (The explanation may be very short, only one line. Just explain how you use those pairs.)
(c) State and prove an equation relating e amd n and d.
(d) Using parts (a), (b), (c), show that if $c \geq 6$ then $d = f = 2$.

Midterm Solutions, Spring 2014, MATH 132

1: Let $S_n = 1^2 + 3^2 + ... + (2n+1)^2$ and $T_n = (n+1)(2n+1)(2n+3)/3$. We shall show, by induction, that $S_n = T_n$ for all natural numbers n. First observe that $S_0 = 1 = T_0$. Now suppose that $n \geq 1$ and that $S_{n-1} = T_{n-1}$. We deduce that $S_n = T_n$ because

$$3(T_n - T_{n-1}) = (2n+1)\big((n+1)(2n+3) - n(2n-1)\big) = 3(2n+1)^2 = 3(S_n - S_{n-1}) \; . \qquad \square$$

2: Part (a). We have $y_{n+2} - 6y_{n+1} + 9y_n = n + 2 - 6(n+1) + 9n = 4n - 4$.

Part (b). The quadratic equation $t^2 - 6t + 9 = 0$ has unique solution $t = 3$. So the general solution for the recurrence relation is $z_n = (A + nC)3^n$ for some A and C. Now $1 = z_0 = A$ and $6 = z_1 = 3(A + C)$, hence $C = 1$. Therefore $z_n = (1 + n)3^n$.

Part (c). Putting $z_n = x_n - y_n$ we see that the sequence $(z_0, z_1, ...)$ satisfies the recurrence relation and the initial conditions in part (b). So $x_n = y + n + z_n = n + (1+n)3^n$.

3: We shall show that such a colouring exists if and only if $1 \leq n \leq 4$. Any such colouring realizes K_n as the disjoint union of two forests, a red forest and a blue forest, each edge of K_n belonging to one or the other of the two forests.

We need only deal with the cases $n = 4$ and $n = 5$ because, if such a colouring exists for K_{n+1}, then such a colouring also exists for K_n. Such a colouring does exist when $n = 4$ because, letting a, b, c, d be the vertices of K_4, we can take ab, bc, cd to be the edges of the red forest. Such a colouring cannot exist for K_5 because a forest with 5 vertices has at most 4 edges, whereas K_5 has 10 edges.

4: Part (a). Let G be a connected graph and let r be the number of vertices with odd degree. Then $r = 0$ if and only if G has an Euler circuit. Also, $r = 2$ if and only if G has an Euler path that is not a circuit.

Part (b). The graph $K_{m,n}$, with $m \leq n$, has an Euler path if and only if m and n are both even or $m = 2$ or $(m, n) = (1, 1)$ or $(m, n) = (1, 2)$.

Part (c). An Euler circuit: $a_1,\ b_2,\ a_4,\ b_1,\ a_3,\ b_4,\ a_2,\ b_3,\ a_1,\ b_1,\ a_2,\ b_2,\ a_3,\ b_3,\ a_4,\ b_4,\ a_1$.

5: Part (a). We have $n - e + f = 2$.

Part (b). The number of such pairs (ϵ, F) is $2e = cf$.

Part (c). As a special case of the Handshaking Lemma, $2e = nd$.

Part (d). By parts (a) and (b), $n - e + 2e/c = 2$. Rearranging and also using part (c), we have $e(c-2)/c = n - 2 = 2e/d - 2$. Since $c \geq 6$, we have $2/3 \leq (c-2)/c < 2/d$. But d is a positive integer and plainly $d \neq 1$. So $d = 2$. From (c), $n = e$ and now, form (a), $f = 2$.

Alternative solutions to question 1: The required equality can be obtained without induction by using the equality $1^2 + 3^2 + ... + (2n+1)^2 = (1^2 + 2^2 + 3^2 + 4^2 + (2n+1)^2)$ or by using the equality $1^2 + 3^2 + ... + (2n+1)^2 = \sum_k (2k+1)^2 = 4\sum_k k^2 + 4\sum_k k + \sum_k 1$ summed over $0 \leq k \leq n$.)

Common Mistakes in Midterm, MATH 132, Spring 2014

General Mistake: *To prove a statement P, it is not enough to show that P implies something that is true.* For example, if P is the statement "$2+2=5$" then, multiplying by 0, we see that P implies the equality $0=0$, which is true. Nevertheless, $2+2$ is not equal to 5.

I subtracted marks from people who argued as above, writing down correct deductions in the wrong order — quite often in Questions 1, 2(a), 5(d) — even though I did understand the underlying thought process. The reason is that your argument ought to be clear to someone who does not already know it. Experienced mathematicians have an aversion to arguments which look like mere checks on consistency. They have lots of experience of how such arguments can often contain subtle mistakes. Actually, experienced mathematicians do sometimes seem to argue backwards, using phrases such as "it suffices to show that". But that skill requires wordpower, and it is beyond the reach of novices who are still having trouble expressing their arguments in complete sentences.

1: The following attempt has some serious defects in commication:

1) $1^2+3^2+...(2n+1)^2=(n+1)(2n+1)(2n+3)/3$
Step 1 for $n=0 \implies 1^2=1.1.3/3=1$
for $n=1 \implies 1^2+3^2=10=2.3.5/3=10$ deals with formula
Step 2 $A=1^2+3^2+...(2n+1)^2=(n+1)(2n+1)(2n+3)/3=A'=4n^3+12n^2+11n+3/3$
$B=1^2+3^2+...(2n+3)^2=(n+2)(2n+3)(2n+5)/3=B'=4n^3+24n^2+47n+30/3$
$A+(2n+3)^2=B \implies B-A=(2n+3)^2$
$B'-A'=(4n^3-24n^2+47n+30)/3-(4n^3-12n^2+11n+3)/3=4n^2+12n+9=(2n+3)^2$
So $B-A=B'-A'$
$1^2+3^2+...(2n+1)^2=(n+1)(2n+1)(2n+3)/3$

Although the candidate has given evidence that he or she can privately check assertions using mathematical induction, that is not what is being tested. The solution is of low value, because it would not communicate much to anyone who cannot already do the question. Only the stronger students in the class would be able to make sense of this argument; and those students could write clearer solutions anyway.

In the line with "Step 2" in it, the sign "=" is used in three completely different ways: the first and third use of "=" is to indicate the *definitions* of A and A'. The second use of "=" has the sense of *to be proved*. The fourth use of "=" indicates a *deduction* about the value of A'. But the mathematical skill that I had to draw upon to glean those interpretations is much greater than the mathematical skill needed simply to do the question.

To express mathematal arguments clearly: *use grammatically understandable sentences*. "We define...", "Assume...", "Therefore...". Incidentally, sentences end with full-stops. No marks are awarded for punctuation. However, if your proof has no full-stops, then that may be a sign that something is amiss.

2: I wonder what the correlation is between people who have been absent from class and people who were unable to do the basic routine bit of this question, part (b)?

3: We applied the usual method for finding Euler paths and Euler circuits. The above Euler circuit for $K_{4,4}$ was obtained by first considering the circuit a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_1. Deleteing the edges of that circuit, the remaining graph was easy to deal with because it is a circuit. There is no need to narrate the method to justify the answer, because the correctness of the answer is easy to check directly.

4: General: A circuit is a particular kind of path. An Euler circuit is a particular kind of Euler path. Despite the emphatic warnings I gave about this during classes, many people wrongly felt that an Euler circuit cannot be an Euler path. I think this is because the Euler Path Theorem is potentially confusing if one does not read it very carefully. Moral: mathematical definitions and theorems do have to be read very carefully!

If you got this wrong, then you lost only one mark in part (a), but you lost more marks in part (b) because you will have missed the main cases where the condition on m and n is satisfied.

Part (a). Many people were unable to express the theorem clearly enough to be unambiguous. For instance, "An Euler path is zero or two vertices with odd-degree". Does that mean "If there is an Euler path then zero or two vertices have odd degree", or does it mean "There is an Euler path if and only if zero or two vertices have odd degree". I subtracted 2 marks when the "if and only if" was absent or unclear.

I subtracted one mark for failing to say that this is a theorem about *graphs*, another mark for failing to mention that the graph must be *connected*.

Part (b). Since the answer is complicated, it is especially important to say clearly what the answer actually is! If you bury the answer in the middle of a complicated justification, that makes it much harder for the reader to extract it.

Part (c) seemed fairly hard to me, though many people were able to do it. My method for finding an Euler path was to follow the procedure that we did in class. I first considered the circuit a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_1. Deleteing the edges of that circuit, the remaining graph was easy to deal with because it is, itself, just a circuit. However, there was no need to narrate the method to justify the answer, since the correctness of the answer is easy to check directly.

5: Part (d). Many people made the above General Mistake, arguing by *assuming* that $d = f = 2$, then showing that this is consistent with the other equalities. But that does no constitute a deductive argument to conclude that $d = f = 2$. After all, the equality $d = f = 2$ is still consistent with the other equalities when we forget the condition that $c \geq 6$. Yet, if we remove the condition $c \geq 6$, then the conclusion $d = f = 2$ can be false: a counter-example is the graph of a cube.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Midterm Makeup, Spring 2014, Bilkent, LJB, 13 May 2014

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum five sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

Question 1: 20% Find real numbers a and b such that

$$1^2 + 4^2 + 7^2 + ... + (3n-2)^2 = 3n^3 + an^2 + bn$$

for all positive integers n. (Do not forget to prove that your answer is correct.)

Question 2: 20% Solve the recurrence relation $x_{n+2} - 4x_{n+1} + 4x_n = 3^n$ with $x_0 = x_1 = 0$. (Hint: Try $x_n = 3^n$. Note, however, that this solution is not correct.)

Question 3: 20% Let G be a graph with n vertices, where $n \geq 10$. Suppose that G has $n-5$ edges, no cycles and no vertices with degree 0? What is the minimum possible number of vertices with degree 1? (You may assume that a tree with m vertices has exactly $m-1$ edges. Any other results about trees must be proved.)

Question 4: 20% Let n be a positive integer. Let C_n be the graph of an n-cube. (Recall that the vertices of C_n are binary strings with length n. Two binary strings have an edge between them provided all except one of their digits are the same.
(a) For which values of n does C_n have an Euler path?
(b) For all of those values of n where there is no Euler path, what is the minimum number of edges that must be added to produce a graph that does have an Euler path?

Question 5: 20% Let G be a planar graph with $n \geq 3$.
(a) Show that G has at most $2n-4$ faces.
(b) Show that it is possible to add edges to G so as to obtain a planar graph with exactly $2n-4$ faces.

Midterm Makeup Answers

The following is just an answer key, not an illustration of a satisfactory exam response.

1: Sum is $(6n^3 - 3n^2 - 1)/2$. That is, $a = -3/2$ and $b = -1/2$.

2: Answer, $x_n = 3^n - (2+n)2^{n-1}$.

3: Answer is 10. (Each of the 5 components is a tree with exactly 2 vertices of degree 1.)

4: For odd n with $n \geq 3$, we must add $2^{n-1} - 1$ edges. (They can join opposite corners.)

5: Use $n - e + f = 2$. Also, for a maximal planar graph, $2e = 3f$.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Final, Spring 2014, Bilkent, LJB, 28 May 2014

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum four sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

Question 1: 5% How many ways are there of putting 6 indistinguishable balls into 3 distinguishable boxes? (Answers are to be explicit, in decimal notation, for example, as 625, not as 5^4. You may use a standard formula, without proof.)

Question 2: 35% Let A and B be finite sets with sizes $|A| = 4$ and $|B| = 6$. (Answers to the following questions are to be explicit, in decimal notation. You may use standard formulas, without proof.)
(a) How many functions $A \to B$ are there?
(b) How many injective functions $A \to B$ are there? (These are sometimes called "into functions".)
(c) How many surjective functions $A \to B$ are there? (These are sometimes called "onto functions".)
(d) How many injective functions $B \to A$ are there?
(e) How many surjective functions $B \to A$ are there?

Question 3: 20% For A and B as in Question 2, let F be the set of functions $A \to B$. Let $\equiv$ be the relation on F such that, given $f, g \in F$, then $f \equiv g$ if and only if there exists a bijection $h : B \to B$ such that $f = h \circ g$.
(a) Show that $\equiv$ is an equivalence relation.
(b) How many equivalence classes does $\equiv$ have?

4: 40% Let E be the encoding function with generating matrix $G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$.
(Yes, two of the columns of G are the same. It is not a misprint.)
(a) Find the encodings $E(w)$ for each message word w.
(b) Find the syndromes for each of the received words

000000 , 000001 , 000010 , 000011 , 000101 , 000110 , 001000 .

(c) Find a received word t such that t together with the 7 received words in (b) can be used as the coset leaders. Explain why only one such received word t exists.
(d) Using those coset leaders, decode the received words 111111 and 110011 and 100001.
(e) For this code, how many errors of transmission can be detected?
(f) Suppose that there was at most error of transmission, in other words, the Hamming distance between the transmitted word $E(w) = q$ and the received word r is $d(q, r) = 1$. Suppose the decoded word was $D(r) = 111$, where D is the decoding function using the above coset leaders. What are the possible values of the original message word w?

Final Exam Solution Key

The following is just a sketch of solutions, not an illustration of a satisfactory exam response.

Marking: Questions 2, 3, 4 as $5+5+5+5+15$, $10+10$, $5+10+5+10+5+5$.

1: $\binom{6+3-1}{2} = 28.$

2: (a) $6^4 = 1296$, (b) $6!/2! = 360$, (c) 0, (d) 0, (e) $4!S(6,4) = 24.65 - 1560$.

3: (a) Routine, (b) Number of equivalence classes is equal to number of equivalence relations on A, which is $S(4,1) + S(4,2) + S(4,3_+ S(4,4) = 1+7+6+1 = 15$.

4: Codewords for 000, 001, 010, 011, 100, 101, 110, 111 are

000000, 001111, 010111, 100001, 110110, 011000, 101110, 111001.

Syndroms of given received words are $000, 001, 010, 011, 101, 110, 111$. Only possibility for t is 000100 because it is the only received word with weight 1 that has the remaining syndrome 100. Letting r denote the received word, σ the syndrome, s the coset leader, $c = r + s = E(w)$, the decodings are as shown.

r	σ	s	c	w
111111	110	000110	111001	111
110011	101	000101	110110	110
100001	000	000000	100001	100

Only one error is detectable.

Finally, the codeword for 111 is 111001. Since all the coset leaders have weight at most 2, the received word must differ from 111001 by at most 2. There is at most one error of transmission, so the correct codeword c must differ from 111001 by at most 3. In other words, the sum $x = c + 111001$ is a codeword with weight at most 3. There are exactly 3 possibilities for x, hence exactly 3 possibilities for $c = x + 111001$. The first three digits of those c are the possible message words, 111 or 011 or 100.

MATH 132: DISCRETE AND COMBINATORIAL MATHEMATICS.

Final Retake, Spring 2014, Bilkent, LJB, 5 June 2014

Time allowed: two hours.
Please put your name and section number on *every* sheet of your manuscript.
Submit one sheet of paper for each question that you attempt (maximum four sheets).
No sheet may have solutions for two separate questions. (Misplaced parts will not be marked).
You may use extra sheets of paper for rough work (which does not need to be handed in).
Do not place sheets of written work in such a way that other candidates can see it.
All use of electronic devices is strictly prohibited. Please remember to silence your phone.
Please DO NOT hand in the question sheet! The examiner already knows the questions.

Question 1: 10% How many ways are there of putting 5 distinguishable balls into 5 indistinguishable boxes? (Some boxes may be left empty.)

Question 2: 30% For a natural number m, let B_m be the number of equivalence relations on a given set with size m. It is to be understood that $B_0 = 1$.
(a) Express B_m in terms of the Stirling numbers (of the second kind).
(b) Evaluate B_1, B_2, B_3, B_4, B_5.
(c) Simplify the expression $\sum_{n=0}^{m} \binom{m}{n} B_n$.

Question 3: 20% A reflexive transitive relation is called a **preorder**. How many preorders are there on the set $\{1, 2, 3\}$? (Do not forget to explain your reasoning clearly.)

4: 40% Consider the coding scheme with generating matrix $G = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$.
(a) Find the corresponding codewords for each message word.
(b) Find the syndromes for each received word of weight 1.
(c) Explain why every set of coset leaders contains all the received words of weight 1.
(d) By considering syndromes, decode 0111110 and 1111100.
(e) For this code, how many errors of transmission can be corrected? How many errors can be detected?
(f) How many possibilities are there for the set of coset leaders (just as a set of received words, not as a sequence of received words).

Solution Key to Final Retake

Question 2 marked as $10 + 10 + 10$, Question 4 as $5 + 10 + 5 + 10 + 5 + 5$.

1: After calculating Stirling numbers $S(m, n)$ using $S(m, n) = S(m-1, n-1) + nS(m-1, n)$ and $S(m, 1) = S(m, m) = 1$, we have $\sum_{n=1}^{5} S(5, n) = 1 + 15 + 25 + 10 + 1 = 52$.

2: **(a)** $B_m = \sum_{n=1}^{5} S(m, n)$.
(b) $(B_1, B_2, B_3, B_4, B_5) = (1, 2, 5, 15, 52)$.
(c) Sum equals B_{m+1} by considering the complement of the equivalence class of a given element.

3: Consider the associated partial ordering on equivalence classes. There are 19, 9, 1 preorders in the case of 3, 2, 1 equivalence classes respectively. So the total number of preorders is $19 + 9 + 1 = 29$.

4: The codewords associated with 000, 001, 010, 011, 100, 101, 110, 111 are

0000000, 0011101, 0101011, 011011, 1000111, 1011010, 1101100, 1110001 .

The syndromes for 0000001, 0000010, ..., 1000000 are the columns of the parity-check matrix, thus, 0001, 0010, 0100, 1000, 1101, 1011, 0111. Letting r denote the received word, σ the syndrome, s the coset leader, $c = r + s = E(w)$, the decodings are as shown.

r	σ	s	c	w
0111110	1000	0001000	0110110	011
1111100	1101	0010000	1101100	110

One error can be corrected, three detected. There are $2^3.3^4.4 = 864$ possible choices for the other coset leaders. (All coset leaders have weight 2 except for the one associated with syndrome 1110. A neat way of dealing with each of the 8 new syndromes is to examine the number of ways of expressing it as a sum of syndromes of received words with weight 1. A longer but more straightforward method is to find the entries in the last 8 lines of the decoding table.)

MATH 110: DISCRETE MATHEMATICS. Midterm 1. Fall 2013

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 5 November 2013, Bilkent University.

1: 30% Let x_0, x_1, x_2, ... be a sequence of real numbers such that $x_{n+2} + x_{n+1} + x_n = 0$ for all natural numbers n. Suppose that $x_0 = 2$ and $x_1 = -1$.

(a) Using mathematical induction, show that $x_n = \begin{cases} 2 & \text{if } n \text{ is divisible by 3,} \\ -1 & \text{if } n \text{ is not divisible by 3.} \end{cases}$

(b) By solving the equation $t^2+t+1 = 0$ and using a standard formula for recurrence relations, give another proof of the conclusion in (a). (Hint: after finding the solutions α and β to the quadratic equation, calculate α^3 and β^3.)

2: 20% Let G be a connected graph (with finitely many vertices). Suppose that each edge of G is coloured either red or blue. Let G_R be the graph with the same vertices as G and with the red edges. Let G_B be defined similarly for the blue edges. (Thus, all three graphs G and G_R and G_B have the same vertices, and every edge of G is either an edge of G_R or else an edge of G_B.) Taking care to be clear about which well-known theorems you are using, show that:

(a) If G_R and G_B both have Euler circuits, then G has an Euler circuit.

(b) If G_R and G both have Euler circuits and G_B is connected, then G_B has an Euler circuit.

3: 25% (a) State and prove a formula relating the number of vertices n, the number of edges e and the number of faces f of a connected planar graph.

(b) Let G and G_R and G_B be as in the Question 2. Now suppose also that G is planar and that G_R and G_B are connected. Let n be the number of vertices. Let f and f_R and f_B be the number of faces of G and G_B and G_R, respectively. Show that $f = n + f_R + f_B - 2$.

4: 25% Let G be a planar graph such that every vertex has degree 3. Suppose that G can be drawn such that every face has exactly 5 edges. Show that G has exactly 20 vertices.

Example solutions to Midterm 1, Fall 2013

There are no “model solutions” to exam questions in mathematics. Often, there is a variety of good arguments, each of which can be succinctly expressed in many different ways.

Solution 1: Part (a). The required formula for x_n plainly holds when $n = 0$ or $n = 1$. Assuming, inductively, that the formula holds for x_n and x_{n+1}, then it plainly holds for x_{n+2}.

Part (b). There exist real numbers A and B such that $x_n = A\alpha^n + B\beta^n$. Solving the quadratic equation, $\alpha = (-1 + i\sqrt{3})/2$ and $\beta = (-1 - i\sqrt{3})/2$. Note that $\alpha^2 = \beta$ and $\beta^2 = \alpha$ and $\alpha^3 = \beta^3 = 1$. By considering the cases $n = 0$ and $n = 1$, we see that $A = B = 1$. The required formula for x_n follows.

Comment: Some candidates did not make it clear what their inductive assumption was. Above, assuming that the formula holds for x_n and x_{n+1}, we deduced that it holds for x_{n+2}. If one were just to assume the formula for x_n, one would not be able to deduce it for x_{n+1}.

Another comment: But another inductive way of doing part (a) is by showing that $x_{n+3} = x_n$, then noting that the formula follows from the cases $n = 0$ and $n = 1$ and $n = 2$.

Yet another comment: As is usual in the first midterm of this introductory course, some candidates incoherently imitated inductive arguments that were given in class. Remember, mathematical induction is just a convenient technique for explaining things. It is not a ritual incantation.

Comment on part (b): As slicker arithmetic, note that $(t^2 + t + 1)(t - 1) = t^3 - 1$, hence $\alpha = e^{2\pi i/3}$ and $\beta = e^{-2\pi i/3}$.

Solution 2: We apply the theorem asserting that a connected graph has an Euler circuit if and only if every vertex has even degree. For a vertex v, let $d(v)$ and $d_R(v)$ and $d_B(v)$ denote the degrees of v in G and G_B and G_R, respectively. Then $d(v) = d_R(v) + d_B(v)$. Part (a): if each $d_R(v)$ and $d_B(v)$ is even, then $d(v)$ is even. Part (b): if each $d(v)$ and $d_R(v)$ is even, then $d_B(v)$ is even.

Comment: Part (a) can also be done by noting that an Euler circuit for G_B followed by an Euler circuit for G_R amounts to an Euler circuit for G.

Solution 3: Part (a). To prove that $n - e + f = 2$, we argue by induction on f. If $f = 1$ then G is a tree and the required conclusion is clear. For $f \geq 2$, we can remove an edge from a circuit, thus obtaining a graph with $n' = n$ vertices, $e' = e - 1$ edges and $f' = f - 1$ faces. Inductively, we may assume that $n' - e' + f' = 2$. Hence $n - e + f = 2$.

Part (b). Let e, e_R, e_B be the number of edges of G, G_R, G_B, respectively. Since all three graphs are planar and connected, $n - e + f = n - e_R + f_R = n - e_B + f_B = 2$. The required equality follows because $e = e_R + e_B$.

Solution 4: We use the formula $n - e + f = 2$. Counting pairs (ϵ, F) where ϵ is an edge of face F, we see that $5f = 2e$. Also, the sum of the degrees of the vertices is $3n = 2e$. Therefore $n(1 - 3/2 + 3/5) = 2$, in other words, $n = 20$.

Comment: As a variant of essentially the same argument, one can use the standard formula $n - 2 = e(c - 2)/c$, where c is the average number of edges per face; in this case, $c = 5$.

MATH 110: DISCRETE MATHEMATICS. Makeup 1. Fall 2013

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 30 December 2013, Bilkent University.

1: Let a, b, c be complex numbers with $a \neq 0$. Let x_0, x_1, x_2, ... be a sequence of complex numbers such that $ax_{n+2} + bx_{n+1} + cx_n = 0$ for all natural numbers n. Suppose that the quadratic equation $at^2 + bt + c = 0$ has two distinct solutions α and β.

(a) 20% Show that there exist complex numbers A and B such that $x_n = A\alpha^n + B\beta^n$.

(b) 10% Show that there does not exist a sequence of complex numbers x_0, x_1, ... satisfying the equalities $x_1 = x_4 = 1$ and $x_{n+2} - \sqrt{2}x_{n+1} + x_n = 0$.

2: 20% Let T be a tree, let r be a natural number, and suppose that T has at least r vertices with degree greater than 2. Show that T has at least $r + 2$ vertices with degree 1. (If you use any standard results, be clear about which results you are using.)

3: Let m be an integer with $m \geq 3$. Let G_m be the graph with $2m$ vertices v_1^1, v_2^1, ..., v_m^1, v_1^2, v_2^2, ..., v_m^2 such that there is an edge between v_i^a and v_j^b if and only if $i \neq j$.

(a) 10% Explain why G_m has an Euler circuit.

(b) 10% Show that, if any two edges are removed from G_m, then the resulting graph has an Euler path.

(c) 10% Give an example of a graph G such that G has an Euler circuit but, if any two edges are removed from G, then the resulting graph does not have an Euler path.

4: 20% Let m and G_m be as in the previous question. For which values of m is G_m planar? (You may use any well-known results about planar graphs, but you must be clear about which results you are using.)

MATH 110: DISCRETE MATHEMATICS. Midterm 2. Fall 2013

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 6 December 2013, Bilkent University.

1: 10% Find a positive integer n such that, given any set N with size $|N| = n$, then the number of reflexive relations on N is equal to the number of symmetric relations on N. (You may use any standard results, provided you are clear about which results you are using.)

2: Let m and n be positive integers with $m \leq n$. Let M and N be sets with sizes $|M| = m$ and $|N| = n$.
(a) 10% How many injective functions $M \to N$ are there? (Explain your reasoning.)
(b) 10% Let $f : M \to N$ and $g : N \to M$ be functions such that $g(f(x)) = x$ for all $x \in M$. Show that f is injective and g is surjective.
(c) 10% Let f be an injective function $M \to N$. How many functions $g : N \to M$ are there such that $g(f(x)) = x$ for all $x \in M$?

3: Let n be any positive integer and let N be a set with size $|N| = n$.
(a) 10% Explain why, for any integer m in the range $0 \leq m \leq n$, the number of subsets of size m in N is equal to the binomial coefficient $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$.
(b) 10% Let $n_1, n_2, ..., n_k$ be natural numbers such that $n = n_1 + n_2 + ... + n_k$. Show that there are exactly $n!/n_1!n_2! \ldots n_k!$ ways of choosing subsets N_1, N_2, ..., N_k such that each $|N_i| = n_i$ and $N_1 \cup N_2 \cup ... \cup N_k = N$.
(c) 10% Give a counter-example to the following statement: *There are exactly* $n!/k!\, n_1! \ldots n_k!$ *equivalence relations* $\equiv$ *on* N *such that* $\equiv$ *has exactly* k *equivalence classes and the equivalence classes have sizes* n_1, n_2, ..., n_k.

4: Let a be any positive integer, let $n = 3a$ and let N be a set with size $|N| = n$.
(a) 10% How many equivalence relations $\equiv$ on N are there such that all the equivalence classes of $\equiv$ have size a?
(b) 10% Let d be the Hamming metric on the set $\mathbb{Z}_2^n$ (the set of binary strings with length n). How many triples (x, y, z) of elements $x, y, z \in \mathbb{Z}_2^n$ are there such that $d(x, y) = d(y, z) = 2a$?
(c) 10% How many of the triples (x, y, z) satisfy $d(x, y) = d(y, z) = d(x, z) = 2a$?

Example solutions to Midterm 2, Fall 2013

Reminder: Of course, there are no "model solutions" to mathematical questions. Often, a conclusion can be justified in many different ways. Always, an argument can be expressed in many different styles.

Solution 1: On N, the number of reflexive relations is $2^{n(n-1)}$. The number of symmetric relations is $2^{n(n+1)/2}$. Those two numbers are equal when $n = 3$.

Comment: The formulas for the numbers of relations can be obtained by considering incidence matrices. The equality $2^{n(n-1)} = 2^{n(n+1)/2}$ holds if and only if $n \in \{0, 3\}$. So $n = 3$ is the unique positive integer solution. None of these observations is required, though.

Solution 2: Part (a). Enumerate $M = \{x_1, ..., x_n\}$. To choose an injection f, there are n choices for $f(x_1)$, then $n-1$ choices for $f(x_2)$, and so on. Finally, there are $n-m+1$ choices for $f(x_m)$. So there are $n(n-1)...(n-m+1) = n!/(n-m)!$ injections $M \to N$.

Part (b). Given $x_1, x_2 \in M$ such that $f(x_1) = f(x_2)$, then $x_1 = g(f(x_1)) = g(f(x_2) = x_2$. So f is injective. Each element $x \in M$ is the image of $f(x)$ under g, so g is surjective.

Part (c). To choose such g, the values at elements of the image of f are already determined but, for each of the $n-m$ elements y of M that are not in the image, there are m choices for $g(y)$. So the number of choices for g is m^{n-m}.

Comment 1: The phrasing for part (c) requires wordpower. A clumsier style, but easier to compose, involves enumerations, as follows. Enumerate $M = \{x_1, ..., x_m\}$ and $N = \{y_1, ..., y_n\}$ such that $f(x_i) = y_i$. To choose g, we must have $g(y_i) = x_i$ when $1 \leq i \leq m$, but there are m choices for $g(y_i)$ when $m+1 \leq i \leq n$. So there are m^{n-m} choices for g.

Comment 2: Parts (b) and (c) were not done well in the exam, undoubtedly because we spent little time on such problems in lectures. Remarkably, several candidates attempted to apply the theory of Stirling numbers of the second kind (unsuccessfully, because that theory is not applicable to this problem). A study of those numbers later in the course and, generally, more practise with combinatorics of abstract functions would seem to be approporiate.

Solution 3: Part (a). To choose, in order, m mutually distinct elements x_1, ..., x_m of N, there are n choices for x_1, then $n-1$ choices for x_2, and so on, finally $n-m+1$ choices for x_m. So there are $n(n-1)...(n-m+1) = n!/(n-m)!$ choices altogether. By the same argument, for each subset M of N with size m, there are $m!$ was of ordering the elements of M. So there are $n!/m!(n-m)!$ choices of M.

Part (b). By part (a), there are $\binom{n}{n_1}$ choices for N_1, then $\binom{m_1}{n_2}$ choices for N_2, generally, $\binom{m_{j-1}}{n_j}$ choices for N_k, where $m_j = n - n_1 - ... - n_j$. We have

$$\binom{n}{n_1}\binom{m_1}{n_2}...\binom{m_{k-2}}{n_{k-1}}\binom{m_{k-1}}{n_k} = \frac{n!}{n_1!m_1!} \cdot \frac{m_1!}{n_2!m_2!} \cdots \frac{m_{k-2}!}{n_{k-1}!m_{k-1}!} \cdot \frac{m_{k-1}!}{n_k!m_k!} = \frac{n!}{n_1!...n_k!} .$$

Part (c). Put $k = 2$ and $n_1 = 1$ and $n_2 = 2$. Then the number of equivalence classes is 3, whereas $n!/k!n_1!...n_k! = 3!/2!1!2! = 3/2$.

Comment 1: An alternative to part (b) would be to argue by induction on k. The case $k = 1$ is trivial, and the case $k = 2$ is part (a). Now suppose that $k \geq 3$ and that the required conclusion holds for all n in the case where k is replaced by $k-1$. Then the number of choices for N_1 is $n!/n_1!(n-n_1)!$, the number of choices for N_2, ..., N_k is $(n-n_1)!/n_2!...n_k!$, hence the number of choices for N_1, ..., N_k is

$$\frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!...n_k!} = \frac{n!}{n_1!...n_k!} \ .$$

This part of the question was intended as an exercise in mathematical induction. I am surprised no-one did it that way. Presumably, some candidates must have noticed that induction can be used, but decided, correctly, that a more direct argument would be just as clear.

Comment 2: It is not hard to show that the formula appearing in part (c) holds if and only if $n_1 = n_2 = ... = n_k$. So the smallest counter-example is the one specified above.

Solution 4: Part (a). As a special case of the formula stated in part (a) of Question 3, the number of ways of choosing the three equivalence classes, N_1, N_2, N_3, in some order, is $(3a)!/(a!)^3$. For each of the equivalence relations $\equiv$, there are $3! = 6$ ways of ordering the three equivalence classes. So there are $(3a)!/3!(a!)^3$ equivalence relations $\equiv$ as specified.

Part (b). We write $x = x_1...x_n$ where each $x_i \in \{0, 1\}$. Recall, $d(x, y) = |\{i : x_i \neq y_i\}|$. There are 2^n choices for y. For each y, the number of choices for x is $\binom{n}{2a}$, because that is the number of ways of choosing the $2a$ indices i for which $x_i \neq y_i$. Similarly, the number of choices for z is $\binom{n}{2a}$. So the number of choices for (x, y, z) is

$$2^n \binom{n}{2a}^2 = 2^{3a}((3a)!/a!(2a)!)^2 \ .$$

Part (c). There are 2^n choices for x. For each x, choosing y and z amounts to choosing the three sets $\{i : x_i \neq y_i = z_i\}$ and $\{i : x_i \neq y_i \neq z_i\}$ and $\{i : x_i = y_i \neq z_i\}$. Each of those three sets has size a, they are mutually disjoint and their union is the whole set of indices i. The argument in part (a) shows that there are $(3a)!/(a!)^3$ ways of choosing those three sets (in order, of course). So the number of choices for (x, y, z) is now

$$2^n \binom{n}{2a} \binom{2a}{a} = 2^{3a} \frac{(3a)!}{(a!)^3} \ .$$

Comment: Part (c) is exceptionally difficult for a first course in combinatorial methods. I would not have been at all surprised if no-one had been able to do it under exam conditions. Congratulations are due to the four candidates who did it successfully. The question is a special case of the following problem: Given any positive integer n then, in the metric space $\mathbb{Z}_2^n$, how many triangles are there with given edge-lengths? For entertainment, I record the following theorem, which supplies a complete answer to that question.

Theorem: *Let n be a positive integer and a, b, c natural numbers. Consider the triples (x, y, z) where $x, y, z \in \mathbb{Z}_2^n$ and $d(x, y) = c$ and $d(x, z) = b$ and $d(y, z) = a$. There exists such a triple if and only if the following conditions hold: each of a, b, c is less than or equal to the sum of the other two; $a + b + c$ is even; $(a + b + c)/2 \leq n$. In that case, the number of such triples is $2^n n!/\alpha!\beta!\gamma!(n - \alpha - \beta - \gamma)!$ where $2\alpha = b + c - a$ and $2\beta = a + c - b$ and $2\gamma = a + b - c$.*

Proof: First suppose that such (x, y, z) exists. Consider the sets

$$A = \{i : y_i \neq z_i\} \; , \qquad B = \{i : x_i \neq z_i\} \; , \qquad C = \{i : x_i \neq y_i\} \; .$$

Since the three elements $x_i, y_i, z_i \in \{0, 1\}$ cannot be mutually distinct, any element of A or B or C must belong to exactly two of those three sets. In other words, $A \cup B \cup C$ is the disjoint union of $B \cap C$ and $A \cap C$ and $A \cap B$. Let $\alpha = |B \cap C|$ and $\beta = |A \cap C|$ and $\gamma = |A \cap B|$. Then

$$a = |A| = \beta + \gamma \; , \qquad b = |B| = \alpha + \gamma \; , \qquad c = |C| = \alpha + \beta \; .$$

Noting that $a + b + c = 2(\alpha + \beta + \gamma)$, we see that a, b, c satisfy the specified existence criterion.

Conversely, suppose that the existence criterion is satisfied. Let $\alpha = (b + c - a)/2$ and $\beta = (a + c - b)/2$ and $\gamma = (a + b - c)/2$, all of which are integers because $a + b + c$ is even. None of a, b, c is greater than the sum of the other two, so α, β, γ are non-negative. Plainly, $\alpha + \beta + \gamma \leq n$. Choosing x, y, z is the same as choosing x, A, B, C. The required conclusion follows because there are 2^n choices for x and, by part (b) of Question 3, there are $n!/\alpha!\beta!\gamma!(n - \alpha - \beta - \gamma)!$ choices for A, B, C. □

MATH 110: DISCRETE MATHEMATICS. Makeup 2. Fall 2013

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 30 December 2013, Bilkent University.

1: 10% A relation $<$ on a set X is said to be **antisymmetric** provided there do not exist elements $x, y \in X$ such that $x < y$ and $y < x$. Suppose that X is a finite set with size n. How many antisymmetric relations on X are there?

2: (a) 10% Give an example of a function $g : Y \to X$ and two functions $f_1, f_2 : X \to Y$ such that $f_1 \circ g = f_2 \circ g = \text{id}_Y$ and $f_1 \neq f_2$. (Recall, the identity function on Y is the function $\text{id}_Y : Y \to Y$ such that $\text{id}_Y(y) = y$ for all $y \in Y$. Thus, the equations $f_1 \circ g = f_2 \circ g = \text{id}_Y$ mean that $f_1(g(y)) = f_2(g(y))$ for all $y \in Y$.)
(b) 10% Give an example of a function $g : Y \to X$ and two functions $h_1, h_2 : X \to Y$ such that $g \circ h_1 = g \circ h_2 = \text{id}_X$ and $h_1 \neq h_2$.
(c) 10% Show that, given a function $g : Y \to X$ and two functions $f, h : X \to Y$ such that $f \circ g = \text{id}_Y$ and $g \circ h = \text{id}_X$ then g is bijective and $f = h = g^{-1}$.

3: Let M and N be finite sets with sizes $m = |M|$ and $n = |N|$. Let X be the set of all functions $M \to N$.
(a) 10% Let $\equiv$ be the relation on X such that, given functions $f, g : M \to N$, then $f \equiv g$ if and only if there exist bijections $u : M \to M$ and $v : N \to N$ satisfying $f = v \circ g \circ u$. Show that $\equiv$ is an equivalence relation.
(b) 10% When $n = 2$, how many equivalence classes does $\equiv$ have?
(c) 10% When $m = 5$ and $n = 3$, how many equivalence classes does $\equiv$ have?

4: Let n be a positive integer. A **ternary string with length** n is a string $x = x_1 x_2 ... x_n$ where each $x_j \in \{0, 1, 2\}$. The set of ternary strings of length n is written as $\mathbb{Z}_3^n$. The **Hamming metric** on $\mathbb{Z}_3^n$ is defined to be the function d such that $d(x, y) = |\{i : x_i \neq y_i\}|$. In other words, the distance between x and y is the number of places i such that the i-th digit of x is different from the i-th digit of y. A **ternary code** of length n is a non-empty subset of $\mathbb{Z}_3^n$. When working with such a code C, the elements of C are called the **codewords**.
(a) 10% Given $x \in \mathbb{Z}_3^n$ and given an integer r such that $0 \leq r \leq n$, how many elements $y \in \mathbb{Z}_3^n$ are there such that $d(x, y) = r$?
(b) 10% Let k be an integer such that $2k + 1 \leq n$ and let C be a code that has as many codewords as possible, subject to the condition that the minimum distance between any two distinct codewords is $2k + 1$. Show that

$$\frac{3^n}{\sum_{r=0}^{2k} 2^r \binom{n}{r}} \leq |C| \leq \frac{3^n}{\sum_{r=0}^{k} 2^r \binom{n}{r}} .$$

(c) 10% For a coding scheme with code C (used in digital communication based on ternary logic), how many errors of transmission can be detected? How many errors of transmission can be corrected? (Briefly justify your answers.)

MATH 110: DISCRETE MATHEMATICS. Final, Fall 2013

Time allowed: two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones and other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question-sheet home.

LJB, 3 January 2014, Bilkent University.

1: 40% Consider the encoding function with generating matrix $G = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$.

(a) 6% Write down the codewords for each of the 8 message words.
(b) 6% Find the parity-check matrix H, and find the syndromes for each of the 8 strings

000000 , 100000 , 010000 , 001000 , 000100 , 000010 , 000001 , 000011 .

(c) 5% Explain why the 8 strings in part (b) can be used as the coset leaders. (Do not forget to explain why the minimal weight condition for selecting coset leaders is satisfied.)
(d) 6% Taking the 8 strings in part (b) as the coset leaders, use the method of calculating syndromes to decode the received words 111111, 011111, 001001.
(e) 6% Write down only the following part of the decoding table: the top line of consisting of the message words, the next line consisting of the codewords, and the last line (which begins with the coset leader 000011).
(f) 5% Instead of 000011, which other strings could be used as the last coset leader?
(g) 6% For this code, how many single-digit errors of transmission can be detected? How many single-digit errors of transmission can be corrected?

2: 30% For positive integers m and n with $m \geq n$, we define the Stirling number $S(m, n)$ be the number of equivalence relations on $\{1, 2, ..., m\}$ that have n equivalence classes. All your arguments in this question must be deduced from that definition. (You may not use any general formulas without proof.)
(a) 10% Show that $S(m + 1, n) = S(m, n - 1) + nS(m, n)$ for all integers $m \geq n \geq 1$.
(b) 5% Briefly explain why $S(m, 1) = 1 = S(m, m)$ for all integers $m \geq 1$.
(c) 5% Using mathematical induction and parts (a) and (b), show that $S(n+1, n) = n(n+1)/2$ for all integers $n \geq 1$.
(d) 5% Directly from the above definition of $S(m, n)$, without using mathematical induction, give another proof that $S(n + 1, n) = n(n + 1)/2$.
(e) 5% Using parts (a) and (b), evaluate $S(7, 4)$.

3: 30% We define a **preorder** on a set X to be a relation $\preceq$ on X such that $\preceq$ is reflexive and transitive.
(a) 10% Let P be the set of preorders on X. We define a relation $\cong$ on P such that, given elements $\preceq_1$ and $\preceq_2$ of P, then $\preceq_1 \cong \preceq_2$ provided there exists a bijection $f : X \to X$ with the property that, for all $x, y \in X$, we have $x \preceq_1 y$ if and only if $f(x) \preceq_2 f(y)$. Show that $\cong$ is an equivalence relation.
(b) 10% Now suppose that $|X| = 3$. How many equivalence classes of preorders on X are there? (Hint: You may find it helpful to describe the equivalence classes by drawing suitable directed graphs. If you do so, you must explain how your diagrams are to be interpreted.)
(c) 10% Still assuming that $|X| = 3$, find the size of each equivalence class. Use that to show that the number of preorders on X is 29. (Little credit will be given for laboriously listing all 29 preorders. That could be done using a computer. The aim of the question is to count the preorders in a systematic and understandable way.)

Example solutions to Final, MATH 110, Fall 2013

Reminder: Of course, there are no "model solutions" in mathematics. Mathematics thrives on diversity of methods and styles.

1: Part (a). Each message word w has codeword wG. The message words 000, 001, 010, 011, 100, 101, 110, 111 have corresponding codewords 000000, 001101, 010111, 011010, 100110, 101011, 110001, 111100.

Part (b), $H = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$, syndromes 000, 110, 111, 101, 100, 010, 001, 011.

Part (d). Decodings are 111, 010, 001. The calculation is expressed in the following table, where r denotes the received word, s the syndrome, v the coset leader, $c = r + v$ the codeword, w the message word.

r	s	v	c	w
111111	011	000011	111100	111
011111	101	001000	010111	010
001001	100	000100	001101	001

Part (e). The specified part of the decoding table is as follows.

message words	000	001	010	011	100	101	110	111
codewords	000000	001101	010111	011010	100110	101011	110001	111100
last line	000011	001110	010100	011001	100101	101000	110010	111111

Part (f). The only other possible strings for the last coset leader are 010100 and 101000. Indeed, all the other strings in the last line have greater weight.

Part (g). The minimal weight of a nonzero codeword is 3, so 2 errors are detectable, 1 error is correctable.

2: Part (a). To choose an equivalence relation on $\{1, ..., m+1\}$ with n equivalence classes, we can either choose an equivalence relation on $\{1, ..., m\}$ with $n-1$ classes, then put $m+1$ in a class of its own, or else we can choose an equivalence relation on $\{1, ..., m\}$ with n classes, then put $m+1$ in one of those classes. In the former case, there are $S(m, n-1)$ choices. In the latter case, bearing in mind that there are n possible classes in which to place $m+1$, there are $nS(m, n)$ choices.

Parts (b). There is only one equivalence relation on $\{1, ..., m\}$ such that all the elements of $\{1, ..., m\}$ are equivalent to each other, and there is only one equivalence relation such that no two distinct elements are equivalent.

Part (c). Let $s(n) = S(n+1, n)$. By parts (a) and (b), $s(1) = 1$ and $s(n) = s(n-1) + n$ for all $n \geq 2$. To prove that $s(n) = n(n+1)/2$, we shall argue by induction on n. The case $n = 1$ is trivial. Now suppose that $n \geq 2$ and that the required formula holds for $s(n-1)$. Thus, $s(n-1) = n(n-1)/2$. We deduce that $s(n) = n(n-1)/2 + n = n(n+1)/2$, as required.

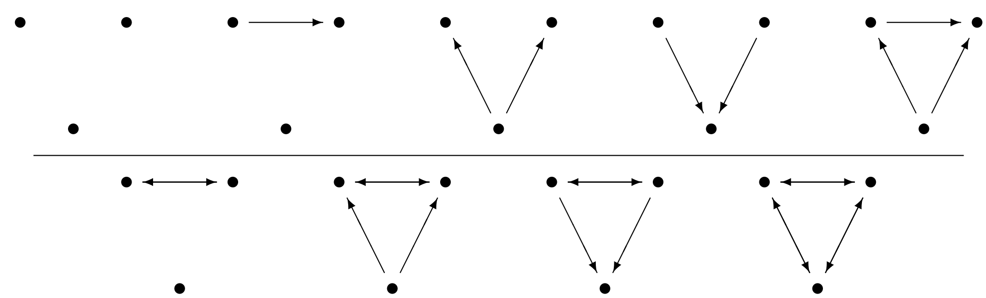
Part (d). The integer $S(n+1, n)$ is the number of equivalence relations such that all the classes have size 1 except for one class of size 2. The number of ways of choosing the class of size 2 is $\binom{n+1}{n} = n(n+1)/2$.

Part (e). By straightforward recursive calculation, $S(7, 4) = 350$.

Comment: A few candidates still had some difficulty with induction. Recall, it is necessary to tell the reader what the inductive assumption is. Otherwise, the reader may not be able to guess what is intended to be implying what. In the above inductive proof in part (c), the inductive assumption was that $n \geq 2$ and that the required formula holds for $s(n-1)$. That is not the only way of organizing the argument. We could, for instance, show that the formula for $s(n)$ implies the formula for $s(n+1)$. One candidate, who did excellently in several other questions, very clearly explained that he or she was showing that the formula for $s(n)$ implies the formula for $s(n-1)$. Of course, that does not work, but it is a correctable slip, and it does earn more credit than arguments which were unclear about what was being assumed and what was being deduced.

3: Part (a). In the notation of the definition of $\cong$, let us call f an **isomorphism** from $\preceq_1$ to $\preceq_2$. Consider elements $\preceq_1, \preceq_2, \preceq_3 \in P$. By considering the identity function on X, we see that $\preceq_1 \cong \preceq_1$. So $\cong$ is reflexive. If $\preceq_1 \cong \preceq_2$, then there is an isomorphism f from $\preceq_1$ to $\preceq_2$, whereupon the inverse f^{-1} is an isomorphism from $\preceq_2$ to $\preceq_1$, hence $\preceq_2 \cong \preceq_1$. We have shown that $\cong$ is symmetric. If $\preceq_1 \cong \preceq_2$ and $\preceq_2 \cong \preceq_3$, then there is an isomorphism f from $\preceq_1$ to $\preceq_2$ and there is an isomorphism g from $\preceq_2$ to $\preceq_2$, whereupon the composite $g \circ f$ is an isomorphism from $\preceq_1$ to $\preceq_3$, hence $\preceq_1 \cong \preceq_3$. We have shown that $\cong$ is transitive.

Part (b). The 9 equivalence classes are indicated in the following diagrams, where an arrow $x \leftarrow y$ indicates that $x \neq y$ and $x \preceq y$.



Part (c). Running through the equivalence classes in the order of the diagrams, the sum of the sizes of the equivalence classes is $1 + 6 + 3 + 3 + 6 + 3 + 3 + 3 + 1 = 29$.