A Discrete Introduction to Conceptual Mathematics

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Chapter 4

Enumerative Combinatorics and Binomial Coefficients

4.1: FISH.

4.2: FISH.

4.3: FISH.

4.4: FISH.

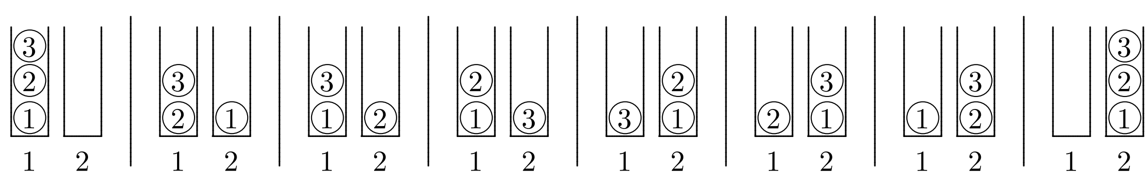
4: Enumerative Combinatorics and binomial coefficients

Questions in enumerative combinatorics often have the form: "How many?"

Of course, questions taking that form occur frequently in most areas of mathematics, but not all such questions would normally be viewed as residing within enumerative combinatorics. Generally, *combinatorics* is a label assigned to areas of mathematics where the mathematical structures involved are not very rich. It is an umbrella term covering conceptually light topics that can be studied without venturing far into heavier branches of mathematics such as algebra, analysis, geometry, topology. That having been said, however, combinatorial problems can be as difficult as problems in any other part of the discipline. Besides, at advanced levels of study, combinatorics does draw substantially from all four of those major branches. Nevertheless, the fact remains that research topics in combinatorics are developments stemming from problems that, at least in original form, are low on structure. Enumerative combinatorics, in particular, arises from questions about sizes of finite sets in contexts that tend to be easily understandable without much knowledge or sophistication.

Consider three differently coloured balls and two differently coloured boxes. For the sake of discussion let us say that one of the balls is red, another green, another blue. Let us say that one of the boxes is white, the other, black. Each ball is to be placed in a box. For the red ball, there are two choices of box, likewise for the blue ball, likewise for the green. Altogether, there are $2^3 = 8$ distinguishable ways of placing the 3 differently coloured balls inside the 2 differently coloured boxes.

The scenario remains essentially unchanged if, instead of supposing the balls and boxes to be distinguishable by colour, we suppose them to be distinguishable by numbering. Let us enumerate the balls and the boxes, calling them ball 1, ball 2, ball 3, box 1, box 2. The 8 distinguishable ways of arranging the balls in the boxes are shown in the following row of 8 diagrams.



Let us generalize that observation.

Coloured Balls in Coloured Boxes Problem: *For positive integers m and n, how many distinguishable ways are there of putting m distinguishable objects into n distinguishable boxes? In other words, how many distinguishable ways are there of putting m differently coloured balls into n differently coloured boxes?*

The solution to the problem is easy.

Solution to the Coloured Balls in Coloured Boxes Problem: *There are exactly n^m ways of putting m differently coloured balls into n differently coloured boxes.*

Easy as the problem may have been, it is not quite obvious. So, following the principle we established in Chapter 1, let us give a very clear deductive explanation.

Proof of the above solution: To choose an arrangement of balls in boxes, we can consider the balls one at a time. For each of the m balls, there are n choices of the box in which the ball is to be placed. Thus, we make a choice m times and, each time we make a choice, there are n possible options. □

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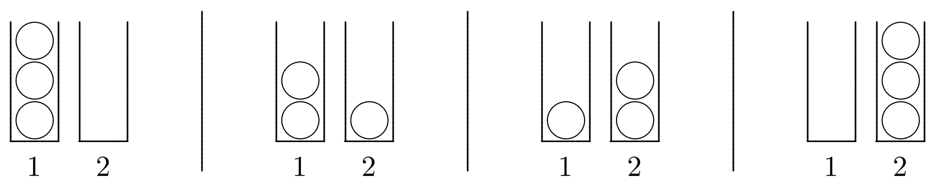
Changing the conditions, let us now suppose that, although we can still distinguish between the boxes, we can no longer distinguish between the balls. The balls all appear to be of exactly the same colour, size, weight, texture, bounciness and so on. How many ways are there now for placing the three balls in the two boxes? The arrangements depicted in the second, third and fourth diagrams above are no longer distinguishable from each other, likewise for the arrangements depicted in the fifth, sixth and seventh diagrams. The answer is 4. The 4 distinguishable kinds of arrangement are as shown in the next row of diagrams.



Again, let us generalize.

Plain Balls in Coloured Boxes Problem: *For positive integers m and n, how many distinguishable ways are there of putting m plain indistinguishable balls into n differently coloured boxes?*

We shall answer that question in the last section of this chapter.

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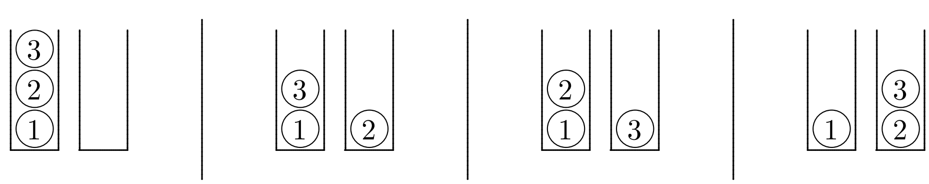
Let us change the conditions again, now supposing that we can distinguish between the balls but not between the boxes. Assumimg that the three balls are differently coloured and that the two boxes are plain and indistinguishable, how many distinguishable ways are there of placing the balls in the boxes? Looking back at the row of eight diagrams above, we see that the answer is 4. Indeed, the arrangements in the first and eighth diagrams are now indistinguishable from each other. In fact, the arrangement in each of those eight diagrams is now indistinguishable from the arrangement in exactly one other of those eight. The next row of diagrams indicates the 4 distinguishable kinds of arrangement. Note that each of the four cases has been illustrated in such a way that the box containing ball 1 is depicted to the left of the other box, but we could equally well have drawn the diagrams with the two boxes appearing the other way around. Since the boxes are indistinguishable, it would not make sense to admit a distinction between a left-hand box and a right-hand box.



It will turn out that, to express the answer in a satisfying way, it is convenient to generalize the problem as follows

Coloured Balls in Plain Boxes Problem: *For positive integers m and n, how many distinguishable ways are there of putting m differently coloured balls into n plain indistinguishable boxes such that no box is left empty?*

Let us write $\sigma(m,n)$ for the answer. Glancing back at the latest row of diagrams, we see that $\sigma(3,1) = 1$ and $\sigma(3,2) = 3$. In our insertion of the clause about no box being left empty, we have not lost sight of the similar problem without that clause. Generally, the number of ways of putting m differently coloured balls into n plain indistinguishable boxes is the sum

$$\sigma(m,1) + \sigma(m,2) + ... + \sigma(m,m) \ .$$

In Section FISH of Chapter FISH, we shall give a formula for $\sigma(m,n)$.

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How many ways are there of putting m plain indistinguishable balls into n plain indistinguishable boxes? How many way are there of doing so such that no box is left empty? Those two problems are much harder. In fact, for arbitrary m and n, neither of those problems has a known answer in the form of an explicit formula. We shall return to these problems when we discuss integer partitions in Section FISH of Chapter FISH.