A Discrete Introduction to Conceptual Mathematics

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Chapter 2

Graph Theory

2: Graph Theory

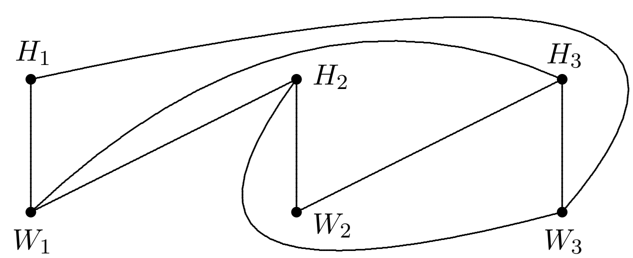
The word *graph* has two main uses in mathematical literature. Both of them are faithful to the etymology of the word, which derives from a classical Greek root which literally meant *to scratch, to scrape* and which also came to mean *to draw, to represent by scratched lines.*

The most popular of the two main uses has the sense of *the graph of a function.* We shall consider that meaning in Chapter 5. The present chapter, though, is concerned with the other sense of the word. Very roughly, we shall understand a *graph* to be something that can represented by a drawing consisting of spots connected by lines. We shall give a precise definition of the term in Section 2.2.

The Three Houses And Three Wells Problem: *Is it possible to connect three houses to three wells by nine footpaths such that each house is connected to a well by a footpath and no two of the footpaths intersect?*

If the problem seems too frivolous, then we could rephrase it as follows. Given microchips A, B, C X, Y, Z, is it possible to connect each of A, B, C to each of X, Y, Z by nine copper data busses all on the same side of a circuit-board?

The diagram below is a graph that has a bearing on the problem. The three spots labelled H_1, H_2, H_3 represent the houses. The three spots labelled W_1, W_2, W_3 represent the wells. The eight lines represent eight non-intersecting footpaths, each footpath connecting a house to a well.



The diagram does not supply a solution to the problem. There is no footpath connecting H_1 to W_2. Plainly, there is no way of adding a footpath between H_1 and W_2 without making an intersection. Nor does the diagram provide a proof that the problem is unsolvable. Conceivably, there could be some other way of connecting H_2 and H_3 to all the wells without any of the wells becoming inaccessible to H_1.

Actually, the problem has no solution. We shall prove this in Section 2.6 using some theory of a particular kind of graph called a *planar graph.*

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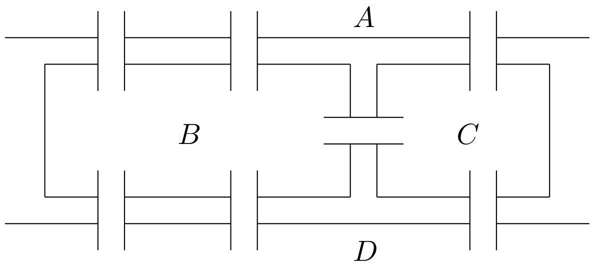
Let us consider an easier problem.

The Königsburg Bridge Problem: *The diagram below is a schematic map of the city of Königsburg, as it was during the middle of the 18th century. A river runs through the city. Part of the city is on one side of the river, region A, part of the city is on the other side, region D, and two other parts of the city are on the islands B and C in the middle of the river. The* *four regions A, B, C, D are connected to each other by seven bridges. Is it possible to make a tour of the city using each bridge exactly once?*



Resolution of the Königsburg Bridge Problem: No. Such a tour of the city is impossible. Suppose, to the contrary, that such a tour is possible. Let X be any one of the regions A, B, C, D. If the tour does not start or finish at X, then the number of bridges used to enter X must be equal to the number of bridges used to exit X. So the number of bridges at X must be even. To put it another way, we have shown that, if the number of bridges at X is odd, then X must be the starting region or the finishing region of the tour. There is only one starting region and only one finishing region, yet all four of the regions A, B, C, D have an odd number of bridges. It is now clear that such a tour cannot exist. □

The idea behind that resolution will be generalized in Sections 2.2 and 2.4.

2.1: Sets as collections of mathematical objects

To define the notion of a graph precisely, so that clear theorems about graphs can be stated and proved, we shall be making use of the notion of a set.

For our purposes in this book, we can understand a **set** to be a collection of objects. When an object x is one of the objects in a set S, we call x an **element** of S and we write $x \in S$. When s is not one of the objects in S, we write $x \notin S$. Given sets S and T, we say that S is **equal** to T, written $S = T$, provided S and T have the same elements.

Given a positive integer n and objects x_1, ..., x_n, the set consisting of precisely those objects is written as $\{x_1, ..., x_n\}$. A set S is said to be **finite** provided S has only finitely many elements, otherwise S is said to be **infinite**. When S is finite, we define the **size** of S, written $|S|$ to be the number of elements in S. The set with no elements, called the **empty set**, denoted by $\emptyset$, is understood to be a finite set with size $|\emptyset| = 0$. The objects x_1, ..., x_n are said to be **mutually distinct** provided $x_i \neq x_j$ whenever $i \neq j$. Thus, x_1, ..., x_n are mutually distinct if and only if $|\{x_1, ..., x_n\}| = n$.

For example, the set of integers x^2 such that x is an integer in the range $-1 \leq x \leq 2$ is

$$\{(-1)^2, 0^2, 1^2, 2^2\} = \{1, 0, 1, 4\} = \{0, 1, 4\} \ .$$

We have $1 \in \{0, 1, 4\}$ and $2 \notin \{0, 1, 4\}$. The set $\{0, 1, 4\}$ is finite and has size $|\{0, 1, 4\}| = 3$. The set of even integers is infinite.

We should mention that the above notion of a set is just a practical working notion that will serve us adequately for the kind of mathematics that we shall be doing. A more sophisticated notion of a set will be sketched in Section 6.FISH.

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FISH pairs and unordered pairs.

Exercise 2.1.A: Let S be finite set. Let $n = |S|$. Express the following in terms of n.
(1) The number of pairs (x, y) where $x, y \in S$, (we mean, $x \in S$ and $y \in S$).
(2) The number of pairs (x, y) where $x, y \in S$ and $x \neq y$.
(3) The number of unordered pairs $\{x, y\}$ where $x, y \in S$.

2.2: The easy half of the Euler Path Theorem

Using the terminology established in the previous section, we shall abstract the essential idea behind our resolution of the Königsburg Bridge Problem.

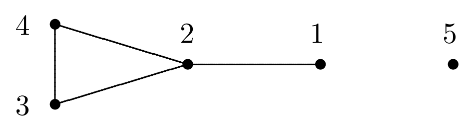
We define a **graph** G to be a pair $G = (V, E)$ where V is a non-empty set and E is a set such that each element of E is an unordered pair of elements of V. The elements of V are called the **vertices** of G. The elements of E are called the **edges**. The two elements x and y of an edge $\epsilon = \{x, y\}$ are called the **end-points** of ϵ. When two vertices x and y are the end-points of an edge, we say that x and y are **adjacent**.

Let us give an example. Consider the graph

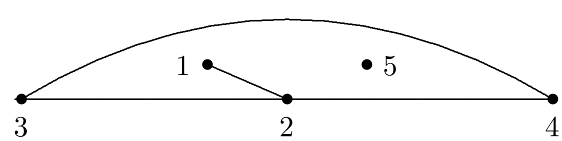
$$(\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}) .$$

The set of vertices of this graph is $\{1, 2, 3, 4, 5\}$. The set of edges is $\{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. The graph has 5 vertices: 1, 2, 3, 4, 5. It has 4 edges: $\{1, 2\}$ and $\{2, 3\}$ and $\{2, 4\}$ and $\{3, 4\}$. The end-points of the edge $\{3, 4\}$ are the vertices 3 and 4. The vertices 3 and 4 are adjacent. The vertices 3 and 1 are not adjacent.

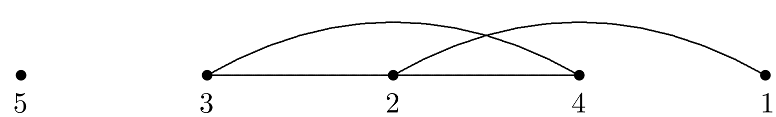
Graphs are often represented by diagrams where the vertices are represented by spots and each edge is represented by a line between its two end-points. One diagram representing the above graph is as follows.



The same graph is also represented by the next diagram.



Let us give one more diagram representing that graph.



Let us point out that there is a distinction between a graph and a diagram representing the graph. The diagram is not the graph itself. Rather, the diagram is something that represents the graph. After all, the three diagrams we drew are all different, yet they all represent the same graph. Analogously, there is a difference between a polar bear and a photograph of a polar bear.

Let us respond to a very natural objection to the above definition of a graph. In the preamble of this chapter, we formed an intuitive notion of a graph. It might be felt that, above, we employed the language of set theory to create something new which does not closely reflect that intuitive notion. Indeed, the set theoretic notion of a graph made little sense until we gave an example to show how to interpret it. At best, it may seem, the set-theoretic notion is something which merely shares some features with that which we ought to have been defining.

But that is, indeed, the best we can do. Capturing intuitive notions is not an easy task. Arguably, set theoretic language actually is a cunning device for precisely defining things which behave just like the elusive notions that we wish to discuss.

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Given a graph G and a natural number n, we define a **path** of **length** n in G to be a sequence of vertices x_0, ..., x_n such that $\{x_{i-1}, x_i\}$ is an edge of G for each i in the range $1 \leq i \leq n$. Such a path is said to be **from** x_0 to x_n.

For example, in the graph depicted above, the sequence 2, 4, 3, 4, 3, 2, 1 is a path of length 6 from 2 to 1.

We define the **degree** $d(x)$ of a vertex x to be the number of edges that have x as an end-point.

In the graph above, the degree of the vertex 3 is $d(3) = 2$. The degree of the vertex 5 is $d(5) = 0$.

Proposition 2.2.1: (Easy Half of the Euler Path Theorem.) *Let s and t be vertices of a finite graph G. Suppose there exists a path from s to t using each edge of G exactly once.*
(1) *If $s = t$ then, for every vertex x of G, the degree of x is even.*
(2) *If $s \neq t$, then the degrees of s and t are odd and the degree of every other vertex of G is even.*

Proof: Consider an path from s to t that uses each edge exactly once. Let x be a vertex such that $x \neq s$ and $x \neq t$. Let δ be the number of times the path enters x. Then δ is also the number of times the path exits x. We have $d(x) = 2\delta$, which is even. If $s = t$, then the number of entries to s is again equal to the number of exits from s, hence $d(s)$ is even. On the other hand, if $s \neq t$, then the number of entries to s is one less than the number of exits from s, hence $d(s)$ is odd and, by a similar argument, $d(t)$ is odd. □

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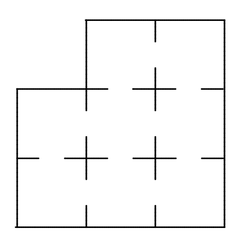
Exercise 2.2.A: A prisoner is trapped in a dungeon consisting of 8 cells and 10 open doorways, as depicted. She wishes to make a tour of the dungeon, passing through each of the ten doorways exactly once.
(1) Is it possible for her to make such a tour?
(2) Is it possible for her to make such a tour that finishes in the cell where she started?



2.3: The Handshaking Lemma

A graph $G = (V, E)$ is said to be **finite** provided the set of vertices V is finite, otherwise G is said to be **infinite**. Plainly, if G is finite, then the set of edges E is finite. In fact, we have the following stronger result.

Remark 2.3.1: *Given a finite graph G, letting n be the number of vertices of G and letting e be the number of edges of G, then $e \leq n(n-1)/2$.*

Proof: The maximum number of edges is attained when any two distinct vertices are adjacent. In that case, the set of edges is the whole set of unordered pairs of vertices. □

The rest of this section is devoted to a relationship between the number of vertices and the degrees of the vertices.

Lemma 2.3.2: (Handshaking Lemma.) *Given a finite graph with e edges and n vertices, enumerating the vertices as x_1, x_2, ..., x_n, then*

$$2e = d(x_1) + d(x_2) + ... + d(x_n) \ .$$

Proof 1: Consider a diagram representing the graph. Cut each edge in half. Plainly, both $2e$ and $d(x_1) + ... + d(x_n)$ are equal to the number of halves of edges. □

Proof 2: Consider the pairs (x, ϵ), where x is one of the end-points of an edge ϵ of the graph. Since each edge ϵ appears in exactly two of those pairs, the number of pairs is $2e$. Since each vertex x appears in exactly $d(x)$ of those pairs, the number of pairs is $d(x_1) + ... + d(x_n)$. □

Employing summation notation, the equality in the lemma can be expressed as

$$2e = \sum_{i=1}^{n} d(x_i) \ .$$

A more elegant and sophisticated way of expressing the sum, avoiding the need to enumerate the vertices is to write

$$2e = \sum_{x \in V} d(x)$$

where V is the set of vertices.

Corollary 2.3.3: *Let G be a finite graph. Let r be the number of vertices x of G such that the degree $d(x)$ is odd. Then r is even.*

Proof: This follows from the latest lemma because, if a sum of integers $d_1 + d_2 + ... + d_n$ is even, then an even number of the terms d_j are odd. □

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Exercise 2.3.A: Give an alternative proof of Remark 2.3.1 using the Handshaking Lemma.

Exercise 2.3.B: Let G be a finite connected graph with 32 edges such that every vertex has degree 4. Find the number of vertices of G.

Exercise 2.3.C: *At a meeting of* 101 *people, each person shakes hands with exactly d of the others. Show that the natural number d must be even.*

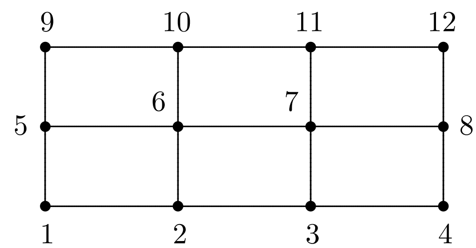
Exercise 2.3.D: Let m and n be natural numbers with $m < n$. Show that mn is even if and only if it is possible for each of n people to shake hands with exactly m of the others.

2.4: Trees

In all the sections of this chapter after the present one, we shall be solving problems about finite graphs by reducing from a given graph to a smaller graph. But a process of reducing to smaller and smaller finite graphs has to cannot continue forever. Eventually, no matter how big and complicated the graph we started with might have been, we are sure to arrive at a graph which cannot reduced any further by the same process. Sometimes, the graph that cannot be reduced any further will be of a particular kind called a *tree*.

Before we can define that notion, we shall be needing some preliminary definitions. A graph G is said to be **connected** provided, for any two vertices x and y of G, there exists a path from x to y in G. Otherwise, G is said to be **disconnected**.

The graph depicted next, with 12 vertices and 20 edges, is connected. The graph we depicted at the beginning of Section 2.2 is disconnected, since it has no path from the vertex 1 to the vertex 5.

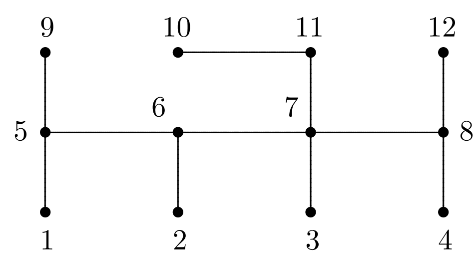


A graph G is said to be **trivial** provided G has only one vertex. Plainly, a trivial graph has no edges. We understand any trivial graph to be connected. That makes sense because, letting x be the vertex, then there is a path of length 0 from x to x.

We define a **circuit** in a graph G to be a path x_0, ..., x_n such that $x_0 = x_n$. We define a **cycle** in G to be a circuit x_0, ..., x_{n-1}, x_0 such that $n \geq 1$ and the vertices x_0, ..., x_{n-1} are mutually distinct

In the graph depicted just above, the path 5, 6, 7, 3, 2, 6, 10, 9, 5 is a circuit of length 8. It is not a cycle since the vertex 6 is repeated. The path 5, 6, 7, 3, 2, 1, 5 is a cycle of length 6.

We define a **tree** to be a connected graph that has no cycle. The next depicted graph is an example of a tree. The trees of concern to us below will be finite trees, that is to say, trees with only finitely many vertices.



Theorem 2.4.1: (Edge-Counting Formula For Trees.) *Let G be a finite connected graph with n vertices and e edges. Then $e \geq n - 1$. Furthermore, G is a tree if and only of $e = n - 1$.*

Proof: Let $m = n - 1$. We shall describe a process that enumerates all the vertices of G as x_0, ..., x_m and, simultaneously, enumerates some of the edges ϵ_1, ..., ϵ_m of G. To commence the process, as step 0, we choose a vertex x_0. If x_0 is the sole vertex of G, then the process halts. Otherwise, as step 1, we choose a vertex x_1 adjacent to x_0, and we let ϵ_1 be the edge $\{x_0, x_1\}$. Generally, after step k, having chosen vertices x_0, ..., x_k and edges ϵ_1, ..., ϵ_k, if there are no more vertices of G then the process halts. Otherwise, as step $k+1$, we choose a vertex x_{k+1} adjacent to some vertex x_i where $0 \leq i \leq k$, and we let $\epsilon_{k+1} = \{x_i, x_{k+1}\}$. Since G is connected, the process continues until, at the end of step m, we have enumerated all the vertices of G. Plainly, the edges ϵ_1, ..., ϵ_m are mutually distinct. Therefore, $e \geq m$.

For each k in the range $0 \leq k \leq m$, let T_k be the graph whose vertices are x_0, ..., x_k and whose edges are ϵ_1, ..., ϵ_k. In view of the way the graphs T_k are constructed, each T_k is connected. Moreover, whenever we introduced a new edge ϵ_k, one of its end-points was a new vertex x_k. So the introduction of ϵ_k cannot create a cycle involving only ϵ_k and the previous edges ϵ_1, ..., ϵ_{k-1}. Therefore, each T_k is a tree.

If $e = m$, then $G = T_m$ and, in particular, G is a tree. Conversely, if G is a tree, then ϵ_1, ..., ϵ_m must constitute all the edges of G, since the introduction of any further edge, say $\{x_i, x_j\}$, would yield a cycle by combining with a path from x_i to x_j in T_m. So, if G is a tree, then $e = m$. □

For any graph G, a vertex of G with degree 1 is called a **leaf** of G. The argument we have given shows that any finite tree has at least one leaf. Indeed, the vertex x_m is a leaf. The next result improves that observation. Note that, of course, any trivial graph is a tree and has exactly one leaf.

Corollary 2.4.2: *Any non-trivial finite tree has at least* 2 *leaves.*

Let T be a non-trivial finite tree. Enumerate the vertices as x_1, ..., x_n. Let e be the number of edges. By the Edge Counting Formula For Trees and the Handshaking Lemma,

$$2n - 2 = 2e = d(x_1) + ... + d(x_n) \ .$$

Since each term $d(x_i)$ is a positive integer, at least two of those terms must be equal to 1. □

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Exercise 2.4.A: Let T be a finite tree with at least one vertex of degree greater than 2. Show that T has at least 3 leaves.

Exercise 2.4.B: Given an example of a tree with no leaves. (Hint: By Corollary 2.4.2, such a tree must be infinite.)

2.5: The Euler Path Theorem

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The following theorem was stated by Euler in an 1736 article in a popular magazine. In the article, Euler proved only the eary half, Proposition 2.2.1. A proof, communicated by Carl Hierholzer to Christian Wiener, was published in an 1873 paper posthumously authored by Hierholzer, actually written by Wiener. The long delay between the statement and the proof should perhaps be interpreted not as evidence of the difficulty of the theorem, but rather as a reflection of the fact that, for a long while, the topic was not taken seriously. Something akin to the modern style of mathematical abstration was just beginning to appear among a few pioneering mathematicians during the 1870s.

Theorem 2.5.1: (Euler Path Theorem.) *Let G be a finite connected graph. Let r be the number of vertices of G that have odd degree. Then G has an Euler path if and only if $r = 0$ or $r = 2$. When $r = 0$, every Euler path for G is an Euler circuit. When $r = 2$, every Euler path for G starts at one of the two vertices of odd degree and ends at the other.*

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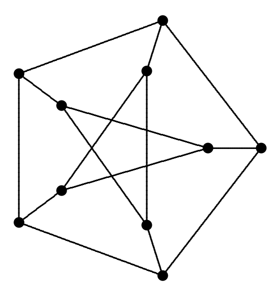
FISH Multigraphs, to accommodate Königsburg Bridge Problem. Directed Multigraphs, illustrate with easy half, defer statement and proof to exercise.

Exercise 2.5.A:

Exercise 2.5.R: Let G be a graph with 10 vertices such that every vertex has degree 3. Suppose that G has a cycle of length 10. Show that G has a cycle with length 3 or 4.

Exercise 2.5.S: The Peterson graph is the graph with 10 vertices and 15 edges shown on the right. (Hint: use the previous exercise.)



(a) Can edges (but no vertices) be added to the Peterson graph to produce a graph with an Euler circuit? If so, what is the minimum number of edges that must be added?

(b) Can edges (but no vertices) be removed from the Peterson graph to produce a connected graph with an Euler circuit? If so, what is the minimum number of edges that must be removed?

2.6: Planar Graphs

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Given a connected graph G, we define a **bridge** of G to be an edge ϵ of G such that the removal of epsilon replaces G with a disconnected graph.

For example, FISH.

Theorem 2.6.1: (Euler's Characteristic Formula.) *Given a finite connected planar graph G with n vertices and e edges, then every planar representation of G has f faces, where*

$$n - e + f = 2 \ .$$

Proof: If $f = 1$, then G is a tree and, by the Edge-Counting Formula for Trees, $e = n - 1$, whence $n - e + f = 2$. For a contradiction, suppose G is a counter-example and that f is minimal. Evidently, $f \geq 2$. So there exists an edge ϵ separating two distinct faces. The edge ϵ cannot be a bridge. Removing ϵ, those two distinct faces become a single face, and we obtain a connected planar graph G' with $n' = n$ vertices, $e' = e - 1$ edges and $f' = f - 1$ faces. But G' has fewer faces than G, so G' cannot be a counter-example, and we must have $n' - e' + f' = 2$. But $n - e + f = n' - e' + f'$. This contradicts the hypothesis that G is a counter-example. □

Again, just as in the proof we gave for the main theorem in the previous section, our argument by contradiction is a little clumsy. As we shall see in Section 3.FISH, it can be recast more smoothly as an argument by mathematical induction.

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Theorem 2.6.2: *Let G be a finite connected planar graph. Suppose G is not a tree. Let n be the number of vertices and e the number of edges of G. Let c be an integer such that $c \geq 3$. Suppose that every cycle in G has length at least c. Then*

$$e \leq (n - 2)c/(c - 2) \ .$$

Proof: Let e' be the number of edges that are not bridges. Consider a planar diagram of G. For that diagram, consider the set of pairs (ϵ, F) where ϵ is a non-bridge edge and F is a face whose boundary includes ϵ. Each non-bridge edge ϵ is part of a cycle, so the face on one side of ϵ must be distinct from the face on the other side. It follows that there are exactly $2e'$ pairs of the specified kind. On the other hand, for each face F, the boundary of F includes at least c non-bridge edges. It follows that the number of pairs is at least cf, where f is the number of faces. We deduce that $2e \geq 2e' \geq cf$. Euler's Characteristic Formula now yields $n - e + 2e/c \geq 2$. So $n - 2 \geq e(1 - 2/c)$. The required inequality follows. □

Theorem 2.6.3: *Given a finite connected planar graph with n vertices and e edges, then $e \leq 1$ or $e \leq 3n - 6$.*

Proof: If G is not a tree then, since every cycle of G has length at least 3, we obtain the required conclusion by putting $c = 3$ and applying the latest theorem.

Now suppose G is a tree. The Edge-Counting Formula For Trees says that $e = n - 1$. We may assume that $e \geq 2$. Hence $n \geq 3$. So $0 < 2n - 5$ and $e < (n-1) + (2n-5) = 3n - 6$. □

Corollary 2.6.4: *Every finite planar graph has a vertex of degree* 5 *or less.*

Proof: Consider a planar graph G. To show that G has a vertex of degree 5 or less, we may assume that G is connected, because otherwise we can replace G with one of its connected components. We may also assume that G has at least 2 edges, otherwise the conclusion is trivial. By the latest theorem, $2e < 6n$, where e is the number of edges, n the number of vertices. Hence, via the Handshaking Lemma, the average of the degrees of the vertices is less than 6. Since the degrees are integers, one of them must be less than or equal to 5. □

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Exercise 2.6.A:

2.7: The Five-Colour Map Theorem

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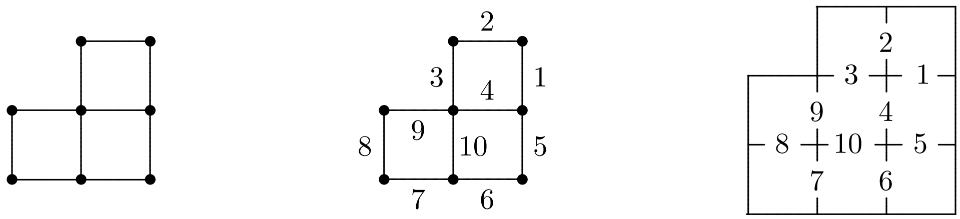
Further Exercises

Exercise 2.A: Let G be a finite connected graph with 16 edges such that every vertex has degree 4.
(1) Find the number of vertices of G.
(2) Now suppose that G is planar. Find the number of faces of G.

Solutions to Exercises

2.1.A:

2.2.A: Part (1). Yes, such a tour is possible. Replacing each cell with a vertex and each doorway with an edge, we obtain the graph depicted below on the right. In the middle diagram, which specifies such a tour, the edges are labelled in the order they are to be traversed. In the right-hand diagram, specifying the same tour, the doorways are labelled in the order they are to be crossed.



Part (2). No, there is no such tour. This follows from the easy half of Euler's path theorem, because two of the vertices of the graph in the left-hand diagram have odd degree.

Alternative argument for part (2). No such tour exists because, if such a tour did exist then, for each cell, the number of times the tour enters the cell would be equal to the number of times the tour exits the cell. That is impossible, because two of the cells have an odd number of doorways. □

Comment: It would be a logical mistake to appeal to the easy half of Euler's Theorem for part (1). Indeed, in Proposition 2.2.1, the existence of the required kind of path is part of the hypothesis, not part of the conclusion. No converse has been established. As it happens, in Section 2.4, we shall prove that the converse does hold. However, since we have not yet obtained that result, we do not yet have any right to make use of it.

2.3.A: The number of edges is maximal when every vertex has degree $n-1$. In that case, $2e = n(n-1)$ by the Handshaking Lemma. □

2.3.B: By the Handshaking Lemma, the number of vertices n and the number of edges e are related by $2e = 4n$. Since $e = 32$, we have $n = 16$.

2.3.C: Consider the graph where the vertices are the people and two vertices are adjacent if and only if they share a handshake. Each vertex has degree d. By the Handshaking Lemma, the number of handshakes e satisfies $2e = 101d$. Therefore d is even. □

2.3.D: Suppose it is possible for the people to shake hands in the specified way. Consider the graph where the vertices are the people and the edges are identified with the handshakes. By the Handshaking Lemma, $2e = mn$ where e is the number of edges. In particular, mn is even.

Conversely, suppose that mn is even. We must show that there exists a graph with n vertices such that every vertex has degree m. Arrange the vertices in an equally spaced way around a circle. If m is even, we can join each vertex to its m nearest vertices. If m is odd then n is even and we can join each vertex to its m furthest vertices. □

2.4.A: Enumerate the vertices x_1, ..., x_n of T such that $d(x_n) \geq 3$. The Edge-Counting

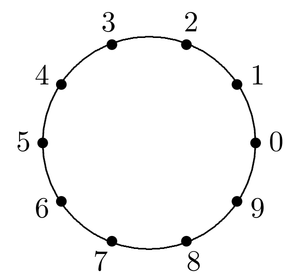
Formula For Trees tells us that the number of edges e satisfies $e = n - 1$. By the Handshaking lemma,

$$2(n-1) - 3 = 2e - 3 \geq d(x_1) + ... + d(x_{n-1}) .$$

Each $d(x_i)$ is a positive integer, so $d(x_i) = 1$ for at least three values of i.

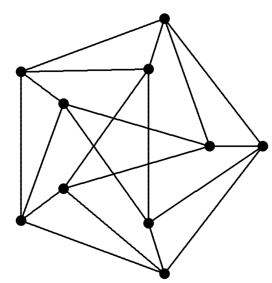
2.4.B: Let T be the tree such that the vertices are the integers and two integers n and m are adjacent if and only if $|n - m| = 1$. Then every vertex of T has degree 2. In particular, T has no leaves.

2.5.R: By the Handshaking Lemma, G has exactly 15 edges. For a contradiction, suppose G has no cycle of length 3 or 4. Let us number the vertices from 0 to 9 such that 0, 1, ..., 9, 0 is a cycle. The diagram on the right depicts G with 5 edges missing. Since every cycle in G has length at least 5, each vertex must be connected, by a missing edge, to the opposite vertex or to one of the neighbours of the opposite vertex. If every vertex were adjacent to its opposite, then there would be a cycle of length 4. So some vertex must be connected, by a missing edge, to one of the neighbours of the opposite vertex. Renumbering the vertices, we may assume that $\{0, 4\}$ is an edge of G. Then the vertex 5 is adjacent to 1 or 9. So there is a cycle of length 4 involving the vertices 0, 4, 5 and either 1 or 9. This is a contradiction, as required.



2.5.S: Part (a). Yes, the minumum number of edges to be added is 5. Indeed, adding 5 edges as depicted, the Euler Path Theorem implies that the new graph has an Euler circuit. If fewer edges are added, then some vertices remain of odd degree and, by the same theorem, there cannot be an Euler circuit.

Part (b). No. For a contradiction, suppose otherwise. Then, by the Euler path theorem, 5 edges must be removed, leaving a connected graph whose every vertex has degree 2. The edges must comprise a circuit with length 10. That contradicts the conclusion of the previous question, since the Peterson graph plainly has no cycle with length less than 5.



2.A: Part (1). By the Handshaking Lemma, the number of vertices is $2.16/4 = 8$.

Part (2). The number of faces is 10. Indeed, letting f be the number of faces, Euler's Characteristic Formula tells us that $8 - 16 + f = 2$.