Some deformations of the fibred biset category

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Abstract

We prove the well-definedness of some deformations of the fibred biset category in characteristic zero. The method is to realize the fibred biset category and the deformations as the invariant parts of some categories whose compositions are given by simpler formulas. Those larger categories are constructed from a partial category of subcharacters by linearizing and introducing a cocycle.

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1 Introduction

One approach to finite group theory involves linear categories whose objects are finite groups. Examples include the biset category studied in Bouc [Bou10], the fibred biset category in Boltje–Coşkun [BC18], the p-permutation category in Ducellier [Duc16] and many subcategories of those. The work behind the present paper has been an attempt, in some cases successful, to characterize such categories in terms of categories that are larger but easier to describe. For the biset category, the theme was initiated in Boltje–Danz [BD13] and developed in [BO]. Our presentation, though, is self-contained and does not presume familiarity with those two papers.

Throughout, we let $\mathcal{G}$ be a non-empty set of finite groups. It is always to be understood that F, G, H, I denote arbitrary elements of $\mathcal{G}$. We let R be a commutative unital ring such that every positive integer has an inverse in R. The inversion condition, expressed differently,

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is that the field of rational numbers $\mathbb{Q}$ embeds in R. We let K be an algebraically closed field of characteristic zero. We let A be a multiplicatively written abelian group.

After reviewing some background in Section 2, we shall introduce the notion of an interior R-linear category $\mathcal{L}$ with set of objects $\mathcal{G}$. Each G acts on the endomorphism algebra $\mathrm{End}_{\mathcal{L}}(G)$ via an algebra map from the group algebra RG. We shall construct a category $\overline{\mathcal{L}}$, called the invariant category of $\mathcal{L}$.

Informally, borrowing a term from algebraic geometry, we call $\mathcal{L}$ a "polarization" of $\overline{\mathcal{L}}$. Let us retain the scare-quotes, because we do not propose a general definition, and we wish only to use the term when the composition for $\mathcal{L}$ is easier to describe than the composition for $\overline{\mathcal{L}}$. A "polarization" of the biset category was introduced in [BD13], and that was extended to some deformations of the biset category in [BO]. In Section 3, as rather a toy illustration, we shall introduce a "polarization" of a K-linear category associated with K-character rings. More substantially, in Section 4, we shall introduce a partial category called the A-subcharacter partial category and, in Section 5, we shall show that a twisted R-linearization of the A-subcharacter partial category serves as a "polarization" of the R-linear A-fibred biset category discussed in Boltje–Coşkun [BC18]. One direction for further study may be towards reassessing the classification, in [BC18], of the simple A-fibred biset functors. We shall comment further on that at the end of the paper.

Also in Section 5, we shall present some deformations of the R-linear A-fibred biset category. To prove the associativity of the deformed composition, we shall make use of the fact that those deformations, too, admit "polarizations" in the form of twisted R-linearizations of the A-subcharacter partial category.

Our hypothesis on R is not significantly more general than the case of an arbitrary field of characteristic zero. Adaptations to other coefficient rings would require further techniques.

2 Interior linear categories

Categories and partial categories arise in our topic mainly as combinatorial structures (in the sense that some familar "up to" qualifications are absent, to wit, all the equivalences of categories below are isomorphisms of categories). Let us organize our notation and terminology accordingly. The idea behind the less standard among the following definitions is not new. It goes back at least as far as Schelp [Sch72]. For clarity, let us present the material in a self-contained way. We define a **partial magma** to be a set $\mathcal{P}$ equipped with a relation $\sim$, called the **matching relation**, together with a function $\mathcal{P} \ni \phi\psi \leftarrow\!\shortmid (\phi, \psi) \in \Gamma(\mathcal{P})$, called the **multiplication**, where $\Gamma(\mathcal{P}) = \{(\phi,\psi) \in \mathcal{P} \times \mathcal{P} : \phi \sim \psi\}$.

We call $\mathcal{P}$ a **partial semigroup** provided the following associativity condition holds: given $\theta, \phi, \psi \in \mathcal{P}$ such that $\theta \sim \phi$ and $\phi \sim \psi$, then $\theta \sim \phi\psi$ if and only if $\theta\phi \sim \psi$, in which case $\theta(\phi\psi) = (\theta\phi)\psi$. When $\theta \sim \phi\psi$, we say that $\theta\phi\psi$ is defined.

Suppose $\mathcal{P}$ is a partial semigroup. An element $\iota \in \mathcal{P}$ satisfying $\iota \sim \iota$ and $\iota^2 = \iota$ is called an **idempotent** of $\mathcal{P}$. Let $\mathcal{X}$ be a set and $\mathcal{I} = (\mathrm{id}_X^{\mathcal{P}} : X \in \mathcal{X})$ a family of idempotents $\mathrm{id}_X^{\mathcal{P}} \in \mathcal{P}$ satisfying the following filtration condition: for all $\phi \in \mathcal{P}$, we have $\mathrm{id}_X^{\mathcal{P}} \sim \phi \sim \mathrm{id}_Y^{\mathcal{P}}$ for unique $X, Y \in \mathcal{X}$, furthermore, $\mathrm{id}_X^{\mathcal{P}}.\phi = \phi = \phi.\iota_Y^{\mathcal{P}}$. We write $\mathrm{cod}(\phi) = X$ and $\mathrm{dom}(\phi) = Y$, which we call the **codomain** and **domain** of ϕ, respectively. Understanding $\mathcal{I}$ to be part of the structural equipment, we call $\mathcal{P}$ a **small partial category** on $\mathcal{X}$, we call an element $\phi \in \mathcal{P}$ a **$\mathcal{P}$-morphism** $\mathrm{cod}(\phi) \leftarrow \mathrm{dom}(\phi)$ and we call $\mathrm{id}_X^{\mathcal{P}}$ the **identity $\mathcal{P}$-morphism** on X. We write

$$\mathcal{P}(X,Y) = \{\phi \in \mathcal{P} : \mathrm{cod}(\phi) = X,\ \mathrm{dom}(\phi) = Y\}$$

and $\mathrm{End}_{\mathcal{P}}(X) = \mathcal{P}(X,X)$. In the context of partial categories, products are called **composites**. Observe that, given $\mathcal{P}$-morphisms ϕ and ψ such that $\phi \sim \psi$, then $\mathrm{dom}(\phi) = \mathrm{cod}(\psi)$. If, conversely, $\phi \sim \psi$ for all $\mathcal{P}$-morphisms ϕ and ψ satisfying $\mathrm{dom}(\phi) = \mathrm{cod}(\psi)$, then we call $\mathcal{P}$ a **small category**. Of course, the latest definition coincides with the usual definition of the same term; a small category in the above sense determines all the structural features of a small category in the conventional sense, and conversely. With suitable formal modifications to the above definitions, the term *small* can be replaced with *locally small*. To preempt logical quibbles, all the categories and partial categories discussed below are to be deemed small, and we shall omit the term *small*, but our discussions can easily be reinterpreted, more generally, for locally small cases.

Given categories $\mathcal{C}$ and $\mathcal{D}$ on a set $\mathcal{X}$, then a functor $\lambda : \mathcal{C} \leftarrow \mathcal{D}$ is said to be **object-identical** provided $\lambda(X) = X$ for all $X \in \mathcal{X}$. Note that, if such λ is an equivalence, then λ is an isomorphism.

Recall, a category is said to be R**-linear** when the morphism sets are R-modules and the composition maps are R-bilinear. Functors between R-linear categories are required to be R-linear on morphisms. We define an **interior** R**-linear category** on $\mathcal{G}$ to be an R-linear category $\mathcal{L}$ on $\mathcal{G}$ equipped with a family (σ_G) of algebra maps

$$\sigma_G : \mathrm{End}_{\mathcal{L}}(G) \leftarrow RG$$

called the **structural maps** of $\mathcal{L}$. We write elements of $F{\times}G$ in the form $f{\times}g$ instead of the conventional (f,g) (because the unconventional notation is the more readable when familiarity has been acquired). We make $\mathcal{L}(F,G)$ become an $R(F{\times}G)$-module such that $f{\times}g$ sends an element $\phi \in \mathcal{L}(F,G)$ to the element

$$^{f\times g}\phi = \sigma_F(f)\,\phi\,\sigma_G(g)^{-1}\ .$$

Proposition 2.1. *Given an interior R-linear category $\mathcal{L}$ on $\mathcal{G}$, then there is an R-linear category $\overline{\mathcal{L}}$ on $\mathcal{G}$ such that, for all $F, G \in \mathcal{G}$, the R-module of $\overline{\mathcal{L}}$-morphisms $F \leftarrow G$ is the $F{\times}G$-fixed R-submodule*

$$\overline{\mathcal{L}}(F,G) = \mathcal{L}(F,G)^{F\times G}$$

and the composition for $\overline{\mathcal{L}}$ is restricted from the composition for $\mathcal{L}$.

Proof. We define $e_G = \dfrac{1}{|G|}\sum_{g\in G} g$ which is an idempotent $Z(RG)$. We have

$$\overline{\mathcal{L}}(F,G) = \sigma_F(e_F)\,\mathcal{L}(F,G)\,\sigma_G(e_G)\ .$$

So $\overline{\mathcal{L}}$ is a category as specified, with identity morphisms $\mathrm{id}_G^{\overline{\mathcal{L}}} = \sigma_G(e_G)$. □

We call $\overline{\mathcal{L}}$ the **invariant category** of $\mathcal{L}$. Note that $\overline{\mathcal{L}}$ need not be a subcategory of $\mathcal{L}$, since $\mathrm{id}_G^{\overline{\mathcal{L}}}$ may be distinct from $\mathrm{id}_G^{\mathcal{L}}$.

We define the R**-linearization** of a partial semigroup $\mathcal{P}$ to be the algebra $R\mathcal{P}$ over R such that $R\mathcal{P}$ is freely generated over R by $\mathcal{P}$ and the multiplication on $R\mathcal{P}$ is given by R-linear extension of the multiplication for $\mathcal{P}$, with the understanding that $\phi\psi = 0$ whenever $\phi \not\sim \psi$. Let $R^\times$ denote the unit group of R. We define a **cocycle** for $\mathcal{P}$ over R to be a function $\gamma : R \leftarrow \mathcal{P}{\times}\mathcal{P}$ satisfying the following two conditions:

Non-degeneracy: Given $\phi, \psi \in \mathcal{P}$, then $\gamma(\phi,\psi) \in R^\times$ if $\phi \sim \psi$, whereas $\gamma(\phi,\psi) = 0$ if $\phi \not\sim \psi$.

Associativity: Given $\theta, \phi, \psi \in \mathcal{P}$ with $\theta\phi\psi$ defined, then $\gamma(\theta,\phi)\gamma(\theta\phi,\psi) = \gamma(\theta,\phi\psi)\gamma(\phi,\psi)$.

Fixing γ, let $R_\gamma\mathcal{P}$ be the R-module freely generated by the set of formal symbols $\{p_\phi : \phi \in \mathcal{P}\}$. We make $R_\gamma\mathcal{P}$ become an (associative, not necessarily unital) algebra over R by taking the multiplication to be such that

$$p_\phi p_\psi = \gamma(\phi,\psi) p_{\phi\psi} \; .$$

We call $R_\gamma\mathcal{P}$ the **twisted linearization** of $\mathcal{P}$ with cocycle γ. When $\gamma(\phi,\psi) = 1$ for all $(\phi,\psi) \in \Gamma(\mathcal{P})$, we call γ the **trivial cocycle** for $\mathcal{P}$. In that case, we have an algebra isomorphism $R_\gamma\mathcal{P} \cong \mathcal{P}$ given by $p_\gamma \leftrightarrow \gamma$.

In later sections, we shall be considering scenarios having the following form. Suppose, now, that $\mathcal{P}$ is a partial category on $\mathcal{G}$. Thus, we are supposing that $\mathcal{P}$ comes equipped with a family of idempotents $(\mathrm{id}_G^{\mathcal{P}})$ satisfying the filtration condition. It is easy to see that the R-linearization $R\mathcal{P}$ is an R-linear category and $\mathrm{id}_G^{R\mathcal{P}} = \mathrm{id}_G^{\mathcal{P}}$. Assume also that $R\mathcal{P}$ is equipped with the structure of an interior R-linear category such that, for all $F, G \in \mathcal{G}$, the action of $F{\times}G$ on $R\mathcal{P}(F,G)$ restricts to an action on $\mathcal{P}(F,G)$. Define

$$\overline{\phi} = \sigma_F(e_F).\phi.\sigma_G(e_G) = \frac{1}{|F|.|G|} \sum_{f\in F, g\in G} {}^f\phi^g$$

for $\phi \in \mathcal{P}(F,G)$. Note that $\overline{\phi} = \overline{{}^f\phi^g}$ and, if we let ϕ run over representatives of the $F{\times}G$-orbits in $\mathcal{P}(F,G)$, then $\overline{\phi}$ runs over the elements of an R-basis for $\overline{R\mathcal{P}}(F,G)$. We have

$$\overline{\phi}.\overline{\psi} = \frac{1}{|G|} \sum_{g\in G} \overline{\phi\,.\,{}^g\psi}$$

for all $\phi \in \mathcal{P}(F,G)$ and $\psi \in \mathcal{P}(G,H)$. Similar comments hold for the twisted linearizations. Let us make those comments, because some modification is needed. Let γ be a cocycle for the partial category $\mathcal{P}$. To confirm that the twisted R-linearization $R_\gamma\mathcal{P}$ is an R-linear category, observe that, writing $\iota = \mathrm{id}_G^{\mathcal{P}}$, then $\gamma(\phi,\iota) = \gamma(\iota,\iota)$, whence

$$\mathrm{id}_G^{R_\gamma\mathcal{P}} = \gamma(\iota,\iota)^{-1} p_\iota \; .$$

Assume now that the structure of an interior R-linear category is imposed on $R_\gamma\mathcal{P}$ instead of $R\mathcal{P}$ and each $\gamma(\phi^g, {}^g\psi) = \gamma(\phi,\psi)$. Again, the elements

$$\overline{p}_\phi = \sigma_F(e_F).p_\phi.\sigma_G(e_G) = \frac{1}{|F|.|G|} \sum_{f\in F, g\in G} {}^f(p_\phi)^g$$

comprise an R-basis for $\overline{R_\gamma\mathcal{P}}(F,G)$. We have

$$\overline{p}_\phi.\overline{p}_\psi = \frac{1}{|G|} \sum_{g\in G} \gamma(\phi, {}^g\psi)\, \overline{p}_{\phi.{}^g\psi} \; .$$

3 The ordinary character category

We shall describe a K-linear category $K\mathcal{A}_K$ associated with ordinary K-character rings of finite groups. We shall then realize $K\mathcal{A}_K$ as the invariant category $\overline{K\mathcal{R}}$ of an interior K-linear category $K\mathcal{R}$.

For a finite group E, we write $\mathcal{A}_K(E)$ to denote the ring of K-characters of E. That is to say, $\mathcal{A}_K(E)$ is the Grothendieck ring of the category of KE-modules. Incidentally, the multiplication on $\mathcal{A}_K(E)$ is given by tensor product over K, but we shall not be making use of that. Given a KE-module M, we identify the isomorphism class of M with the K-character $\chi_K : K \leftarrow E$ of M. Thus, $\mathcal{A}_K(E)$ has a basis consisting of the irreducible K-characters of E. The K-linear extension $K\mathcal{A}_K(E)$ can be identified with the K-module of class functions $K \leftarrow E$.

Any KF-KG-bimodule M can be regarded as a $K(F{\times}G)$-module by writing $fmg^{-1} = (f{\times}g)m$ for $f \in F$, $g \in G$, $m \in M$. In particular, the isomorphism class of M can be identified with the K-character $\chi_M : K \leftarrow F{\times}G$. We form a K-linear category $K\mathcal{A}_K$ with morphism K-modules $K\mathcal{A}_K(F,G) = K\mathcal{A}_K(F{\times}G)$ and with composition $K\mathcal{A}_K(F,H) \leftarrow K\mathcal{A}_K(F,G) \times K\mathcal{A}(G,H)$ such that, writing $Z = X \otimes_{KG} Y$ for a KF-KG-bimodule X and a KG-KH-bimodule Y, then $\chi_Z = \chi_X.\chi_Y$. We call $K\mathcal{A}_K$ the **K-character category** or the **ordinary character category** on $\mathcal{G}$, (up to changes of coefficients and Galois automorphisms, $K\mathcal{A}_K$ is independent of K). The next result describes the composition more explicitly.

Lemma 3.1. *Let $\xi \in K\mathcal{A}(F,G)$ and $\eta \in K\mathcal{A}(G,H)$. Let $f \in F$ and $h \in H$. Then*

$$(\xi.\eta)(f{\times}h) = \frac{1}{|G|}\sum_{g\in G}\xi(f{\times}g)\eta(g{\times}h)\ .$$

Proof. Let X, Y, Z be as above. By K-linearity, we may assume that $\xi = \chi_X$ and $\eta = \chi_Y$. Let $\zeta = \chi_Z$. Then $\zeta(f{\times}h) = (\xi.\eta)(f{\times}h)$. Let $\widehat{Z} = X \otimes_K Y$ regarded as a module of $K(F{\times}G{\times}H)$ such that

$$(f{\times}g{\times}h)(x \otimes_K y) = fxg^{-1} \otimes_K gyh^{-1} = (f{\times}g)x \otimes_K (g{\times}h)y$$

for $g \in G$, $x \in X$, $y \in Y$. Then $\chi_{\widehat{Z}}(f{\times}g{\times}h) = \xi(f{\times}g)\eta(g{\times}h)$. The required equality follows because $Z \cong \widehat{Z}/\mathrm{span}_K\{xg^{-1} \otimes_K y - x \otimes_K gy\} \cong e_G.\widehat{Z}$. □

Let $\mathcal{R}$ be the partial category on $\mathcal{G}$ such that $\mathcal{R}(F,G) = F{\times}G$ and, given $u{\times}v \in \mathcal{R}(F{\times}G)$ and $v'{\times}w \in \mathcal{R}(H{\times}G)$, then $(u{\times}v) \sim (v'{\times}w)$ if and only if $v = v'$, furthermore, $(u{\times}v)(v{\times}w) = u{\times}w$. We make the K-linearization $K\mathcal{R}$ become an interior K-linear category by defining

$$\sigma_G(g) = \sum_{v\in G}({}^g v){\times}v$$

where ${}^g v = gvg^{-1}$. Thus, $F{\times}G$ acts on $\mathcal{R}(F,G)$ by ${}^f(u{\times}v)^g = ({}^f u){\times}(v^g)$, where $v^g = g^{-1}vg$. Let

$$\mu_{F,G} : \overline{K\mathcal{R}}(F,G) \leftarrow K\mathcal{A}_K(F,G)$$

be the K-linear isomorphism given by

$$\mu_{F,G}(\xi) = \sum_{u\in F, v\in G}\xi(u{\times}v)\,\overline{u{\times}v}\ .$$

Proposition 3.2. *The maps $\mu_{F,G}$, for $F, G \in \mathcal{G}$, determine an object-identical isomorphism of K-linear categories $\mu : \overline{K\mathcal{R}} \leftarrow K\mathcal{A}_K$.*

Proof. Let u, v, v', w be as above. Write $[v]_G$ for the G-conjugacy class of v. By comments in Section 2, $\overline{u{\times}v}.\overline{v'{\times}w} = 0$ unless $[v]_G = [v']_G$, in which case, $\overline{u{\times}v}.\overline{v'{\times}w} = \overline{u{\times}w}/|[v]_G|$. Direct calculation now yields $\mu_{F,G}(\xi).\mu_{G,H}(\eta) = \mu_{F,H}(\xi.\eta)$ for all ξ and η as in Lemma 3.1. □

4 The subcharacter partial category

We shall introduce a category $\mathcal{S}^A$ on $\mathcal{G}$, called the A-**subcharacter partial category** on $\mathcal{G}$. We shall construct a twisted R-linearization $R_\ell\mathcal{S}^A$ of $\mathcal{S}^A$ parameterized by a multiplicative monoid homomorphism $\ell : R^\times \leftarrow \mathbb{N} - \{0\}$. After equipping $R_\ell\mathcal{S}^A$ with structural maps to make $\underline{R_\ell\mathcal{S}^A}$ become an interior R-linear algebra, we shall explicitly describe the invariant category $\overline{R_\ell\mathcal{S}^A}$. That description will be applied to deformations of the R-linear A-fibred biset category in the next section. Some of our terminology and notation is adapted from [Bar04], [BO] and Boltje–Coşkun [BC18], but our account is self-contained.

To introduce some notation that we shall be needing, let us review the definition of the subgroup category $\mathcal{S}$ on $\mathcal{G}$. (The category would be written as $\mathcal{S}_\mathcal{G}$ in the notation of [BO].) Consider the groups $F, G, H, I \in \mathcal{G}$. We let $\mathcal{S}(F,G)$ denote the set of subgroups of $F{\times}G$. Let $U \in \mathcal{S}(F,G)$, $V \in \mathcal{S}(G,H)$, $W \in \mathcal{S}(H,I)$. We define

$$\Gamma(U,V) = \{f{\times}g{\times}h : f{\times}g \in U, g{\times}h \in V\} .$$

After Bouc [Bou10, 2.3.19], we define the **star product** $U * V \in \mathcal{S}(F,H)$ to be

$$U * V = \{f{\times}h : f{\times}g{\times}h \in \Gamma(U,V)\} .$$

Plainly, $*$ is associative. We point out that, defining

$$\Gamma(U,V,W) = \{f{\times}g{\times}h{\times}i : f{\times}g \in U, g{\times}h \in V, h{\times}i \in W\}$$

then $U * V * W = \{f{\times}i : f{\times}g{\times}h{\times}i \in \Gamma(U,V,W)\}$. We make $\mathcal{S}$ become a category by taking the composition to be star product.

Below, when we have established the construction of the partial category $\mathcal{S}^A$, it will be clear that $\mathcal{S}^A$ coincides with $\mathcal{S}$ when A is trivial. First, though, we need the patience for a few definitions. As a subgroups of G, we define

$$U^\bullet = \{g \in G : 1{\times}g \in U\} , \quad \Gamma_\cap(U,V) = \{g \in G : 1{\times}g{\times}1 \in \Gamma(U,V)\} , \quad {}^\bullet V = \{g \in G : g{\times}1 \in V\} .$$

The next lemma is part of Boltje–Danz [BD13, 3.5], the lemma after that, [BO, 9.2].

Lemma 4.1. (Boltje–Danz.) *With the notation above,*

$$|\Gamma_\cap(U,V)|.|\Gamma_\cap(U*V,W)| = |\Gamma_\cap(U,V*W)|.|\Gamma_\cap(V,W)| .$$

Lemma 4.2. *With the notation above,* $|U|.|V| = |U^\bullet.{}^\bullet V|.|\Gamma_\cap(U,V)|.|U*V|$.

For a finite group E, we define an A-**character** of E to be a homomorphism $A \leftarrow E$. We define an A-**subcharacter** to be a pair (T,τ) consisting of a subgroup T of E and an A-character τ of E. The set $\mathcal{S}^A(E)$ of A-subcharacters of E becomes an E-set via the actions of E on the two coordinates. When E is understood from the context, we write $[T,\tau]$ to denote the E-orbit of (T,τ). We write $\mathcal{S}^A[E]$ to denote the set of E-orbits in $\mathcal{S}^A(E)$.

We define $\mathcal{S}^A(F,G) = \mathcal{S}^A(F{\times}G)$ and $\mathcal{S}^A[F,G] = \mathcal{S}^A[F{\times}G]$. Let (U,μ), (V,ν), (W,ω) be A-subcharacters in $\mathcal{S}^A(F,G)$, $\mathcal{S}^A(G,H)$, $\mathcal{S}^A(H,I)$, respectively. We write $(U,\mu) \sim (V,\nu)$ provided $\mu(1{\times}g)\nu(g{\times}1) = 1$ for all $g \in \Gamma_\cap(U,V)$. When that condition holds, we define $\mu * \nu$ to be the A-character of $U * V$ given by $(\mu * \nu)(f{\times}h) = \mu(f,g)\nu(g,h)$ for $f{\times}g{\times}h \in \Gamma(U,V)$.

Proposition 4.3. *Defining composition by* $(U,\mu) * (V,\nu) = (U * V, \mu * \nu)$ *when* $(U,\mu) \sim (V,\nu)$*, then* $\mathcal{S}^A$ *becomes a partial category.*

Proof. The main work is in showing that the conditions

- that $(U,\mu)\sim(V,\nu)$ and $(U*V,\mu*\nu)\sim(W,\omega)$,
- that $(V,\nu)\sim(W,\omega)$ and $(U,\mu)\sim(V*W,\nu*\omega)$,

are equivalent and, when they hold, $(\mu*\nu)*\omega=\mu*(\nu*\omega)$. It is straightforward to confirm that the two conditions are equivalent to:

- for all $g{\times}h\in G{\times}H$ satisfying $1{\times}g{\times}h{\times}1\in\Gamma(U,V,W)$, we have $\mu(1{\times}g)\nu(g{\times}h)\omega(h{\times}1)=1$.

Plainly, when the three equivalent conditions hold, the expression $\mu*\nu*\omega$ is unambiguous and

$$(\mu*\nu*\omega)(f{\times}i)=\mu(f{\times}g)\nu(g{\times}h)\omega(h{\times}i)$$

for all $f{\times}g{\times}h{\times}i\in\Gamma(U,V,W)$. The main work is done. To finish the proof, we observe that $\mathrm{id}_G^{\mathcal{S}^A}=s^{G,G}_{\Delta(G),1}$ where $\Delta(G)=\{y{\times}y:y\in G\}$ and 1 denotes the trivial A-character. □

By Lemma 4.1 there is a cocycle γ_ℓ for $\mathcal{S}^A$ given by

$$\gamma_\ell((U,\mu),(V,\nu))=\ell(|\Gamma_\cap(U,V)|)$$

when $(U,\mu)\sim(V,\nu)$. We define $R_\ell\mathcal{S}^A=R_{\gamma_\ell}\mathcal{S}^A$. Thus,

$$R_\ell\mathcal{S}^A(F,G)=\bigoplus_{(U,\mu)\in\mathcal{S}^A(F,G)}R\,s^{F,G}_{U,\mu}$$

as a direct sum of regular R-modules, where $s^{F,G}_{U,\mu}$ is a formal symbol and

$$s^{F,G}_{U,\mu}s^{G,H}_{V,\nu}=\begin{cases}\ell(|\Gamma_\cap(U,V)|)\,s^{F,H}_{U*V,\mu*\nu} & \text{if } (U,\mu)\sim(V,\nu),\\ 0 & \text{otherwise.}\end{cases}$$

Specializing some observations made in Section 2, we note that the element

$$\overline{s}^{F,G}_{U,\mu}=\sigma_F(e_F).s^{F,G}_{U,\mu}.\sigma_G(e_G)$$

depends only on F, G and the $F{\times}G$-orbit $[U,\mu]$ of (U,μ). We have

$$\overline{R_\ell\mathcal{S}^A}(F,G)=\bigoplus_{[U,\mu]\in\mathcal{S}^A[F,G]}R\,\overline{s}^{F,G}_{U,\mu}\;.$$

To complete an explicit description of the category $\overline{R_\ell\mathcal{S}^A}$, we now supply a formula for the composition. By viewing $\mathcal{S}^A(F,G)$ as an F-G-biset, the notation in the equation ${}^g(V,\nu)={}^{g\times1}(V,\nu)$ makes sense for any $g\in G$, similarly for the notation ${}^gV$ and ${}^g\nu$.

Theorem 4.4. *Let $F,G,H\in\mathcal{G}$. Let $[U,\mu]\in\mathcal{S}^A[F,G]$ and $[V,\nu]\in\mathcal{S}^A[G,H]$. Then*

$$\frac{\overline{s}^{F,G}_{U,\mu}}{|U|}\cdot\frac{\overline{s}^{G,H}_{V,\nu}}{|V|}=\frac{1}{|G|}\sum_g\frac{\ell(|\Gamma(U,{}^gV)|)}{|\Gamma(U,{}^gV)|}\cdot\frac{\overline{s}^{F,H}_{U*{}^gV,\mu*{}^g\nu}}{|U*{}^gV|}$$

where g runs over representatives of the double cosets $U^\bullet.g.{}^\bullet V\subseteq G$ such that $(U,\mu)\sim{}^g(V,\nu)$.

Proof. By the last line of Section 2,

$$\overline{s}^{F,G}_{U,\mu}.\overline{s}^{G,H}_{V,\nu} = \frac{1}{|G|}\sum_y \ell(|\Gamma(U,{}^yV)|).\overline{s}^{F,H}_{U*{}^yV,\mu*{}^y\nu}$$

where $\gamma(y) = |\Gamma(U,{}^yV)|$ and y runs over those elements of G such that $(U,\mu) \sim {}^y(V,\nu)$. We have $(U,\mu) \sim {}^{y'}(V,\nu)$ and $\gamma(y) = \gamma(y')$ for all $y' \in U^\bullet.y.{}^\bullet V$. So

$$\overline{s}^{F,G}_{U,\mu}.\overline{s}^{G,H}_{V,\nu} = \frac{1}{|G|}\sum_g |U^\bullet.g.{}^\bullet V|.\ell(|\Gamma(U,{}^gV)|).\overline{s}^{F,H}_{U*{}^gV,\mu*{}^g\nu}\,.$$

Since $|U^\bullet.g.{}^\bullet V| = |U^\bullet.{}^\bullet({}^gV)|$ and $|{}^gV| = |V|$, Lemma 4.2 yields the required equality. □

5 The fibred biset category

We shall review the notion of the **R-linear A-fibred biset category** $R\mathcal{B}^A$ on $\mathcal{G}$. Then we shall introduce, more generally, an R-linear category $R_\ell\mathcal{B}^A$ on $\mathcal{G}$. To confirm the associativity of the composition for $R_\ell\mathcal{B}^A$, we shall apply Theorem 4.4.

A discussion $R\mathcal{B}^A$, including an interpretation as the R-linear extension of a Grothendieck ring, can be found in Boltje–Coşkun [BC18, Sections 1, 2]. We shall work with the following characterization of $R\mathcal{B}^A$. The morphism R-modules are

$$R\mathcal{B}^A(F,G) = \bigoplus_{[U,\mu]\in\mathcal{S}^A[F,G]} R[(F{\times}G)/(U,\mu)]$$

where, for our purposes, we can regard $[(F{\times}G)/(U,\mu)]$ as a formal symbol uniquely determined by the $F{\times}G$-orbit $[U,\mu]$. See [BC18, Section 1] for an interpretation, not needed below, of $[(F\times G)/(U,\mu)]$ as the isomorphism class of an A-fibred biset $(F{\times}G)/(U,\mu)$. The composition for $R\mathcal{B}^A$ is given by

$$[(F{\times}G)/(U,\mu)].[(G{\times}H)/(V,\nu)] = \sum_g [(F{\times}H)/(U*{}^gV,\mu*{}^g\nu)]$$

where g runs as in Theorem 4.4. It is easy to check that the right-hand side of the formula is well-defined, independently of the choices of double coset representatives g and orbit representatives (U,μ) and (V,ν). The associativity of the composition follows from [BC18, 2.2, 2.5] or, alternatively, Theorem 5.1 below. The identity $R\mathcal{B}^A$-morphism on G is $[(G{\times}G)/(\Delta(G),1)]$.

Generalizing, we define

$$R_\ell\mathcal{B}^A(F,G) = \bigoplus_{[U,\mu]\in\mathcal{S}^A[F,G]} R\,d^{F,G}_{U,\mu}$$

where $d^{F,G}_{U,\mu}$ is a formal symbol uniquely determined by F, G and $[U,\mu]$. We make $R_\ell\mathcal{B}^A$ become an R-linear category on $\mathcal{G}$ by defining the composition to be such that

$$d^{F,G}_{U,\mu}.d^{G,H}_{V,\nu} = \sum_g \frac{\ell(|\Gamma(U,{}^gV)|)}{|\Gamma(U,{}^gV)|}\,d^{F,H}_{U*{}^gV,\mu*{}^g\nu}$$

again with g running as in Theorem 4.4. In a moment, to confirm that $R_\ell\mathcal{B}^A$ is an R-linear category, we shall make use of the "polarization" $R_\ell\mathcal{S}^A$. We let

$$\nu_{F,G} : \overline{R_\ell\mathcal{S}^A}(F,G) \leftarrow R_\ell\mathcal{B}^A(F,G)$$

be the R-linear isomorphism given by $\nu_{F,G}(d^{F,G}_{U,\mu}) = |G|\,\overline{s}^{F,G}_{U,\mu}\,/\,|U|$.

Theorem 5.1. *The composition for $R_\ell \mathcal{B}^A$ is associative and $R_\ell \mathcal{B}^A$ is an R-linear category on $\mathcal{G}$. The maps $\nu_{F,G}$, for $F, G \in \mathcal{G}$, determine an object-identical isomorphism of R-linear categories $\nu : \overline{R_\ell \mathcal{S}^A} \leftarrow R_\ell \mathcal{B}^A$.*

Proof. Theorem 4.4 implies that $\nu_{F,G}(d^{F,G}_{U,\mu}).\nu_{G,H}(d^{G,H}_{V,\nu}) = \nu_{F,H}(d^{F,G}_{U,\mu}.d^{G,H}_{V,\nu})$. By R-linearity, the composition is associative. The identity $R_\ell \mathcal{B}^A$-morphism on G is $d^{G,G}_{\Delta(G),1}$. □

We have the following immediate corollary, realizing $R\mathcal{B}^A$ as the invariant category not of $R\mathcal{S}^A$ but of a deformation of $R\mathcal{S}^A$.

Corollary 5.2. *Suppose $\ell(n) = n$ for all positive integers n. Then there is an object-identical isomorphism of R-linear categories $\overline{R_\ell \mathcal{S}^A} \cong R\mathcal{B}^A$ given by $|U| d^{F,G}_{U,\mu} \leftrightarrow |G|.\overline{[(F \times G)/(U,\mu)]}$.*

At the time of writing, we do not know whether an analogue of the generic semisimplicity result [BO, 1.1] holds for the twisted K-linearization $K_\ell \mathcal{S}^A$ of $\mathcal{S}^\mathcal{A}$. An approach to directly adapting the argument in [BO] would be to make use of an analogue of [BC18, 3.7] that follows quickly from Green's result in [BO, 2.1]. More speculatively, if such a generic semisimplicity result does hold, then it might have a bearing on the problem of classifying the simple $K_\ell \mathcal{S}^A$-modules and, from there, via Theorem 5.1, the problem of classifying the simple $K_\ell \mathcal{B}^A$-modules.

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