Semisimplicity of some deformations of the subgroup category and the biset category

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Abstract

We introduce some deformations of the biset category and prove a semisimplicity property. We also consider another group category, called the subgroup category, whose morphisms are subgroups of direct products, the composition being star product. For some deformations of the subgroup category, too, we prove a semisimplicity property. The method is to embed the deformations of the biset category into the more easily described deformations of the subgroup group category.

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1 Introduction

This paper concerns two categories and some of their deformations. One of those two can be defined immediately without any specialist prerequisites. We define the **subgroup category** $\mathcal{S}$ as follows. The class of objects of $\mathcal{S}$ is the class of groups. Consider groups R, S, T. Let us write the elements of the direct product $R{\times}S$ in the form $r{\times}s$ instead of (r,s). We define the set of $\mathcal{S}$-morphisms $R \leftarrow S$ to be the set of subgroups of $R{\times}S$. Given subgroups $U \leq R{\times}S$ and $V \leq S{\times}T$, we define the $\mathcal{S}$-composite of U and V, denoted $U * V$, to be the subgroup of $R{\times}T$ consisting of those elements $r{\times}t$ such that there exists $s \in S$ satisfying $r{\times}s \in U$ and

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$s{\times}t \in V$. The operation $*$, called the **star product**, is familiar in the theory of bisets, as in Bouc [Bou10, 2.3.19], for instance.

We mention that a category constructed on similar lines, with finite sets as objects, has been shown to admit rich theory by Bouc–Thévenaz [BT] and citations therein.

Preliminaries aside, the groups under consideration will be finite. We let $\mathcal{K}$ be any non-empty set of finite groups. For some of the categories we shall be considering, the objects are arbitrary finite groups. To preempt logical objections when applying ring theoretic techniques, we shall often be passing from a large category to a small full subcategory whose set of objects is $\mathcal{K}$. However, up to equivalence of categories, the dependence on $\mathcal{K}$ will be only on the group isomorphism classes occurring in $\mathcal{K}$. So our rigorous constraint on expression will impose little constraint on interpretation.

We let $\mathbb{K}$ be a field of characteristic zero. The role of $\mathbb{K}$ will be as a coefficient ring. Some easy generalizations to other coefficient rings do not seem to merit tergiversation over hypotheses.

The scenario under investigation will be determined by $\mathcal{K}$ and $\mathbb{K}$, together with one other item of data, namely, a monoid homomorphism ℓ to the unit group $\mathbb{K}^\times = \mathbb{K} - \{0\}$ from the monoid of positive integers $\mathbb{N}-\{0\}$. Let Π denote the set of primes. Then ℓ is determined by the family $(\ell(q) : q \in \Pi)$ of arbitrary elements $\ell(q) \in \mathbb{K}^\times$. We call ℓ **algebraically independent** with respect to $\mathcal{K}$ when, letting q run over the prime divisors of the orders of the groups in $\mathcal{K}$, the $\ell(q)$ are algebraically independent over the minimal subfield $\mathbb{Q}$ of $\mathbb{K}$. The role of ℓ will be to parameterize some deformations of some categories.

Let $\mathcal{S}_\mathcal{K}$ denote the full subcategory of $\mathcal{S}$ such that the objects of $\mathcal{S}_\mathcal{K}$ are the elements of $\mathcal{K}$. In Section 2, we shall construct an algebra Λ over $\mathbb{K}$. The morphisms in $\mathcal{S}_\mathcal{K}$ index a $\mathbb{K}$-basis for Λ, called the square basis. In terms of the square basis, the multiplication operation derives from the composition operation in the most straightforward imaginable way, except for the introduction of a cocycle that depends on ℓ. Thus, employing some terminology that will be defined in that section, Λ is a twisted category algebra of $\mathcal{S}_\mathcal{K}$.

We can view Λ as a deformation of the category algebra $\mathbb{K}\mathcal{S}_\mathcal{K}$ of $\mathcal{S}_\mathcal{K}$. Our reason for mentioning the less specific notion of a deformation is that, in Section 9, we shall be making use of Λ to examine an algebra Γ that can be viewed as a deformation of the category $\mathbb{K}\mathcal{B}_\mathcal{K}$ obtained from the biset category $\mathcal{B}$ by extending coefficients to $\mathbb{K}$ and confining the objects to $\mathcal{K}$. (To prevent misunderstanding, we mention that, in the notation of Section 2, the category algebra of $\mathcal{B}_\mathcal{K}$ over $\mathbb{K}$ could also be written as $\mathbb{K}\mathcal{B}_\mathcal{K}$, but we shall not be considering that category algebra in this paper.)

Corollary 6.5 says that local semisimplicity never holds for the category algebra $\mathbb{K}\mathcal{S}_\mathcal{K}$, except in trivial cases. Nevertheless, our main result, proved in Section 7, is that Λ is semisimple in the following generic sense.

Theorem 1.1. *If ℓ is algebraically independent with respect to $\mathcal{K}$, then Λ is locally semisimple.*

Surprisingly, for arbitrary $\mathcal{K}$, $\mathbb{K}$, ℓ, it is not hard to classify the simple Λ-modules. That is in contrast to the situation for $\mathbb{K}B_\mathcal{K}$, where a suitable closure condition has to be imposed on $\mathcal{K}$ to avoid the "vanishing problem" discussed in Rognerud [Rog19]. In Section 3, we shall show that, for Λ, the "vanishing problem" itself vanishes. That will allow us, in Theorems 3.5 and 3.6, to give two descriptions of a classification of simple Λ-modules for arbitrary $\mathcal{K}$.

When $\mathcal{K}$ is finite, Λ is unital. For unital rings, local semisimplicity is just semisimplicity. In Section 4, we shall prove the following implication of that condition. Given a positive integer n, we write Mat_n to indicate an algebra of $n{\times}n$ matrices. Given a group R, we write $\mathrm{Aut}(R)$ to denote the automorphism group of R in the category of groups.

Theorem 1.2. *Suppose $\mathcal{K}$ is finite and Λ is semisimple. Let E run over representatives of the isomorphism classes of the factor groups of the elements of $\mathcal{K}$. Then we have an algebra isomorphism*

$$\Lambda \cong \bigoplus_{E} \mathrm{Mat}_{n_E}(\mathbb{K}\mathrm{Aut}(E))$$

where n_E is the number of triples (G, B, Y) such that $\mathcal{K} \ni G \geq B \trianglerighteq Y$ and $B/Y \cong E$.

In Section 5, to prepare for a deeper study, we review some results and techniques from Boltje–Danz [BD13]. We express the material as a passage to another $\mathbb{K}$-basis for Λ, the round basis, which lacks the closure property of the square basis but instead has the advantage that products vanish except under strong conditions. After applying those techniques to the general case in Sections 6 and 7, we shall consider, in Section 8, a particular simple Λ-module called the trivial Λ-module. We shall give criteria for the projectivity of the trivial Λ-module. We shall see that, for finite $\mathcal{K}$, projectivity of the trivial Λ-module is equivalent to simplicity of the associated block algebra.

Theorem 9.3 describes an embedding of Γ in Λ. That will yield the following corollary.

Corollary 1.3. *If Λ is locally semisimple, then Γ is locally semisimple.*

For one particular case of ℓ, we have $\Gamma \cong \mathbb{K}\mathcal{B}_{\mathcal{K}}$. A theorem of Serge Bouc, appearing in [Bar08, 1.1], asserts that $\mathbb{K}\mathcal{B}_{\mathcal{K}}$ is locally semisimple if and only if every group in $\mathcal{K}$ is cyclic. In Section 9, we shall prove the following generalization of half of that result.

Theorem 1.4. *If every element of $\mathcal{K}$ is cyclic, then the algebra $\Lambda \cong \Gamma$ is locally semisimple.*

The aim of this paper is to initiate study in a speculative direction which, to yield applications, may require generalizations. In [BO], the $\mathbb{K}$-linear biset category $\mathbb{K}\mathcal{B}$ is replaced by the $\mathbb{K}$-linear C-fibred biset category $\mathbb{K}\mathcal{B}_C$, where C is a supercyclic group. Replacement of $\mathbb{K}\mathcal{B}$ with the p-permutation category $\mathbb{K}\mathcal{T}$, studied in Ducellier [Duc16], might be difficult but interesting, since p-blocks that are p-permutation equivalent in the sense of Boltje-Xu [BX08] correspond to associate idempotents of small full subcategories of $\mathbb{K}\mathcal{T}$.

One possible application of semisimple deformations of $\mathbb{K}\mathcal{B}$ may be in the study of those functors $\mathbb{K}$–mod $\leftarrow \mathbb{K}\mathcal{B}$ that admit suitable deformations. The same may pertain to $\mathbb{K}\mathcal{B}_C$ and $\mathbb{K}\mathcal{T}$, if those two categories can be shown to admit deformations with semisimplicity properties.

To indicate another possible line of further study, let us suppose that $\mathbb{K}$ arises as the field of fractions of a complete local noetherian ring $\mathbb{O}$ whose residue field $\mathbb{F}$ has prime characteristic p. Given an algebra A over $\mathbb{O}$ such that A is freely and finitely generated over $\mathbb{O}$ and the $\mathbb{K}$-linear extension of A is semisimple, then the $\mathbb{F}$-linear reduction of A admits a theory of decomposition numbers and a factorization of the Cartan matrix. A paradigmatic case is that where A is the group algebra of a finite group. Another case, concerning Mackey categories, is discussed in Thévenaz–Webb [TW95]. It might seem absurd to suggest that such a decomposition theory might be applied in contexts involving $\mathbb{K}\mathcal{B}$, $\mathbb{K}\mathcal{B}_C$, $\mathbb{K}\mathcal{T}$. After all, those three categories lack the prerequisite semisimplicity property. But the suggestion may cease to seem absurd when we consider the possibility of reinstating semisimplicity by passing to suitable deformations.

2 Cocycle deformation of the subgroup category

After setting up some notation and terminology, we shall define the algebra Λ mentioned above, and we shall classify the simple Λ-modules.

We do not require rings to be unital. Even when working with unital rings, we do not require subrings to be common-unity subrings. Given a ring A, we define a **corner subring** of A to be a subring B such that $B \geq BAB$. We call a ring monomorphism $\nu : A \leftarrow C$ a **corner embedding** provided $\nu(C)$ is a corner subring. We call A **locally unital** provided every finite subset of A is contained in a unital corner subring of A, we mean, a subring having the form eAe where e is an idempotent of A. We consider A-modules only when A is locally unital and, in that case, we require that every element of an A-module is fixed by an idempotent of A. The next result follows easily from the special case in Green [Gre07, 6.2g].

Proposition 2.1. (Green.) *Let B be a corner subring of a locally unital ring A. Then B is locally unital. Furthermore, the condition $T \cong BS$ characterizes a bijective correspondence $[T] \leftrightarrow [S]$ between the isomorphism classes $[T]$ of simple B-modules and the isomorphism classes of those simple A-modules S satisfying $BS \neq 0$.*

For any property $\mathfrak{P}$ of unital rings such that $\mathfrak{P}$ is closed under passage to corner subrings, a ring A is said to be **locally** $\mathfrak{P}$ provided A is locally unital and $\mathfrak{P}$ holds for every unital corner subring of A. We shall be especially concerned with the condition of local semisimplicity. (The common practice of using *semisimple* to mean *locally semisimple* may be harmless, since it does not change the meaning of *semisimple* in the established context of unital rings. Nevertheless, we adopt the longer term because it carries a cautionary reminder of the generality.) Before we depart from abstract ring theory, let us record a lemma for later use.

Lemma 2.2. *Let A be a locally artinian ring. Let B be a corner subring of A. Let S be a simple A-module. Define $T = BS$, which is a simple B-module by the above proposition. Then B is locally artinian and we have an isomorphism of division rings* $\mathrm{End}_B(T) \cong \mathrm{End}_A(S)$.

Proof. Plainly, B is locally artinian. The specified action plainly yields an embedding of division rings $\nu : \mathrm{End}_B(T) \leftarrow \mathrm{End}_A(S)$. Let i be a primitive idempotent of B such that $iT \neq 0$. Then Bi and Ai are projective covers of T and S, respectively. So $\mathrm{End}_B(T) \cong iBi/J(iBi)$ and $\mathrm{End}_A(S) \cong iAi/J(iAi)$. But $iBi = iAi$. □

We understand an **algebra** over $\mathbb{K}$ to be a ring equipped with a compatible $\mathbb{K}$-module structure. That is to say, algebras over $\mathbb{K}$ are to be associative but not necesarily unital.

We deem all categories to be locally small. Given objects X and Y of a category $\mathcal{C}$, we write $\mathcal{C}(X,Y)$ to denote the set of $\mathcal{C}$-morphisms $X \leftarrow Y$. It is to be understood that a morphism determines its domain and codomain, in other words, the morphism sets $\mathcal{C}(X,Y)$ are mutually disjoint. We write $\mathrm{id}_X^{\mathcal{C}}$ or id_X to denote the identity $\mathcal{C}$-morphism of X. When $\mathcal{C}$ is small, we write $\mathrm{mor}(\mathcal{C})$ for the set of morphisms of $\mathcal{C}$, we write $\mathrm{obj}(\mathcal{C})$ for the set of objects of $\mathcal{C}$, and we call $\mathcal{C}$ a category on $\mathrm{obj}(\mathcal{C})$. For arbitrary $\mathcal{C}$ and a set $\mathcal{O}$ of objects of $\mathcal{C}$, we write $\mathcal{C}_{\mathcal{O}}$ for the full subcategory of $\mathcal{C}$ on $\mathcal{O}$.

Recall, a category is said to be $\mathbb{K}$**-linear** when the morphism sets are $\mathbb{K}$-modules and the composition maps are $\mathbb{K}$-bilinear. When $\mathcal{C}$ is small and $\mathbb{K}$-linear, we define the **algebra associated with** $\mathcal{C}$ to be the algebra

$$\mathcal{C}_{\mathrm{alg}} = \bigoplus_{X,Y \in \mathrm{obj}(\mathcal{C})} \mathcal{C}(X,Y)$$

with multiplication given by composition, the product of two incompatible morphisms being zero. Systematically, in this paper, we shall employ the language of category theory when working with $\mathbb{K}$-linear categories that are possibly large, but we shall shift to the richer language of ring theory when working with small $\mathbb{K}$-linear categories. When $\mathcal{C}$ is small and $\mathbb{K}$-linear, all the features of $\mathcal{C}$ can be recovered from the algebra $\mathcal{C}_{\mathrm{alg}}$ together with the complete family of mutually orthogonal idempotents $(\mathrm{id}_X^{\mathcal{C}} : X \in \mathrm{obj}(\mathcal{C}))$. For instance, the morphism $\mathbb{K}$-modules can be recovered from the equality $\mathcal{C}(X,Y) = \mathrm{id}_X^{\mathcal{C}}.\mathcal{C}.\mathrm{id}_Y^{\mathcal{C}}$. We shall write $\mathcal{C}$ instead of $\mathcal{C}_{\mathrm{alg}}$, relying on context to resolve any ambiguity. To diminish or eliminate even any potential for ambiguity, we shall work freely with the following alternative definition which, at least in the context of our ring theoretic approach, is equivalent to the definition above: a small $\mathbb{K}$-linear category is an algebra $\mathcal{C}$ over $\mathbb{K}$ equipped with a complete family of mutually orthogonal idempotents whose indexing set, denoted $\mathrm{obj}(\mathcal{C})$, is called the set of objects of $\mathcal{C}$.

For small $\mathcal{C}$, given a subset $\mathcal{O} \subseteq \mathrm{obj}(\mathcal{C})$, then $\mathcal{C}_{\mathcal{O}}$ is a corner subalgebra of $\mathcal{C}$. Note that $\mathcal{C}_{\mathcal{O}}$ is unital if and only if $\mathcal{O}$ is finite. Proof of the next remark is easy.

Remark 2.3. *Let $\mathcal{C}$ be a small $\mathbb{K}$-linear category. Then the following three conditions are equivalent: $\mathcal{C}$ is locally semisimple; every full subcategory of $\mathcal{C}$ is locally semisimple; $\mathcal{C}_{\mathcal{O}}$ is semisimple for every finite set $\mathcal{O}$ of objects of $\mathcal{C}$.*

Given $\mathcal{C}$ as in the remark, a $\mathcal{C}$-module M and $X \in \mathrm{obj}(\mathcal{C})$, we define $M(X) = \mathrm{id}_X.M$, which we regard as a module of the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(X) = \mathcal{C}(X,X) = \mathrm{id}_X^{\mathcal{C}}.\mathcal{C}.\mathrm{id}_X^{\mathcal{C}}$. We mention that, in a well-known manner, M can be viewed as a functor to the category of $\mathbb{K}$-modules, and $M(X)$ can be regarded as the evaluation at X. But we shall not be making use of that interpretation.

As another preliminary, let us say a few words on category algebras and twisted category algebras. The following constructions are already discussed in Linckelmann [Lin04], so our coverage is brief. Let $\mathcal{C}$ be any category. We define, as follows, a $\mathbb{K}$-linear category $\mathbb{K}\mathcal{C}$ called the **$\mathbb{K}$-linearization** of $\mathcal{C}$. The objects of $\mathbb{K}\mathcal{C}$ are the objects of $\mathcal{C}$. Given objects X and Y, then $\mathbb{K}\mathcal{C}(X,Y)$ is the $\mathbb{K}$-module freely generated by $\mathcal{C}(X,Y)$. The composition for $\mathbb{K}\mathcal{C}$ is obtained from the composition for $\mathcal{C}$ by $\mathbb{K}$-linear extension. We have $\mathrm{id}_X^{\mathbb{K}\mathcal{C}} = \mathrm{id}_X^{\mathcal{C}}$. When $\mathcal{C}$ is small, the $\mathbb{K}$-linearization $\mathbb{K}\mathcal{C}$ is small, and we can pass to the algebra $\mathbb{K}\mathcal{C}$, which we call the **category algebra** of $\mathcal{C}$ over $\mathbb{K}$. As an equivalent definition, for small $\mathcal{C}$, the algebra $\mathbb{K}\mathcal{C}$ is the algebra over $\mathbb{K}$ such that $\mathbb{K}\mathcal{C}$ has $\mathbb{K}$-basis $\mathrm{mor}(\mathcal{C})$ and the multiplication operation on $\mathbb{K}\mathcal{C}$ is the $\mathbb{K}$-linear extension of the composition operation, again with the product of two incompatible morphisms taken to be zero.

For any category $\mathcal{C}$, a **cocycle** for $\mathcal{C}$ over $\mathbb{K}$ is defined to be a formal family of functions

$$\gamma_{X,Y,Z} : \mathbb{K}^{\times} \leftarrow \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z)$$

indexed by triples of objects X, Y, Z of $\mathcal{C}$, such that, dropping the subscripts,

$$\gamma(\theta,\phi)\gamma(\theta \circ \phi, \psi) = \gamma(\theta, \phi \circ \psi)\gamma(\phi,\psi)$$

for all $\mathcal{C}$-morphisms θ, ϕ, ψ with $\theta \circ \phi \circ \psi$ defined. We define the **twisted category** associated with γ to be the $\mathbb{K}$-linear category $\mathbb{K}_\gamma\mathcal{C}$ such that $\mathbb{K}_\gamma\mathcal{C} = \mathbb{K}\mathcal{C}$ as $\mathbb{K}$-modules and the composition $\circ_\gamma$ satisfies

$$\phi \circ_\gamma \psi = \gamma(\phi,\psi)\, \phi \circ \psi$$

for all $\mathcal{C}$-morphisms ϕ and ψ such that $\phi \circ \psi$ is defined. The associativity of the composition is clear. It is easy to check that the identity $\mathbb{K}_\gamma\mathcal{C}$- morphism on X is $\gamma(\mathrm{id}_X, \mathrm{id}_X)^{-1}\mathrm{id}_X^{\mathcal{C}}$. When $\mathcal{C}$ is small, the algebra $\mathbb{K}_\gamma\mathcal{C}$ is called the **twisted category algebra** associated with γ.

We now turn to the subgroup category $\mathcal{S}$. For any group R, we write $\mathcal{S}(R)$ to denote the set of subgroups of R. Goursat's Theorem, well-known and easy to prove, provides a classification of the subgroups of a direct product of two groups.

Proposition 2.4. (Goursat's Theorem.) *Let R and S be groups. Then the condition*

$$U = \{x{}_\bullet U \times yU_\bullet : x \times y \in {}^\bullet U \times U^\bullet, x{}_\bullet U = \theta_U(yU_\bullet)\}$$

characterizes a bijective correspondence $U \leftrightarrow ({}^\bullet U, {}_\bullet U, \theta_U, U_\bullet, U^\bullet)$ between the subgroups U of $R{\times}S$ and the quintuples $({}^\bullet U, {}_\bullet U, \theta_U, U_\bullet, U^\bullet)$ such that $R \geq {}^\bullet U \trianglerighteq {}_\bullet U$ and $U_\bullet \trianglelefteq U^\bullet \leq S$ and θ_U is an isomorphism ${}^\bullet U/{}_\bullet U \leftarrow U^\bullet/U_\bullet$.

Of course, the five parameters ${}^\bullet U$, ${}_\bullet U$, θ_U, $U_\bullet$, $U^\bullet$ depend on R and S as well as U. When we apply the proposition to elements U of the set $\mathcal{S}(R,S) = \mathcal{S}(R{\times}S)$, we shall usually be understanding the codomain R and the domain S to be implicit in the specification of U. To guarantee disjointness of morphism sets, a prerequisite condition in the above definition of a category algebra, we can understand the elements of $\mathcal{S}(R,S)$ to be triples having the form (R,U,S) where $U \leq R{\times}S$. But let us not include that in our notation. In the scenario of the proposition, we write

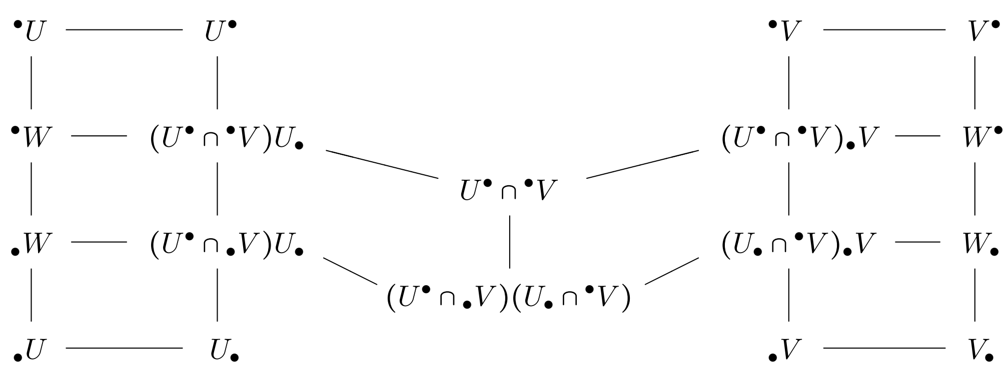
$$U = \Delta({}^\bullet U, {}_\bullet U, \theta_U, U_\bullet, U^\bullet) \ .$$

We abridge the notation in some special cases, as follows. Let $A \leq R$ and $B \leq S$. Given an isomorphism $\theta : A \leftarrow B$, we write $\Delta(\theta) = \Delta(A, 1, \theta, 1, B)$, which makes sense upon identifying A with $A/1$ and similarly for B. Given $R \geq C \leq S$, we write $\Delta(C) = \Delta(\mathrm{id}_C) = \{(c,c) : c \in C\}$. In particular, the identity $\mathcal{S}$-morphism on R is $\mathrm{id}_R^{\mathcal{S}} = \Delta(R)$.

The following description of the star product, though familiar to experts on bisets, is worth briefly reviewing in the notation that we shall be using. For another account of the star product, presented in the context of bisets, see Bouc [Bou10, Chapter 2]. Let R, S, T be groups. Let $U \in \mathcal{S}(R,S)$ and $V = \mathcal{S}(S,T)$. Write $W = U * V$. A straightforward application of Zassenhaus' Butterfly Lemma yields isomorphisms

$$\frac{{}^\bullet W}{{}_\bullet W} \cong \frac{(U^\bullet \cap {}^\bullet V)U_\bullet}{(U^\bullet \cap {}_\bullet V)U_\bullet} \cong \frac{U^\bullet \cap {}^\bullet V}{(U^\bullet \cap {}_\bullet V)(U_\bullet \cap {}^\bullet V)} \cong \frac{(U^\bullet \cap {}^\bullet V){}_\bullet V}{(U_\bullet \cap {}^\bullet V){}_\bullet V} \cong \frac{W^\bullet}{W_\bullet} \ .$$

The isomorphisms are the canonical isomorphisms expressed in the following variant of the diagram that gives the Butterfly Lemma its name. The four horizontal lines on the left indicate how, via ϕ, four subgroups of ${}^\bullet U$ containing ${}_\bullet U$ correspond to four subgroups of $U^\bullet$ containing $U_\bullet$. A similar comment applies to the four horizontal lines on the right, with ψ in place of ϕ.



The next proposition repeats the above description of W in a more explicit way.

Proposition 2.5. *With the notation above, write $M = U^\bullet \cap {}^\bullet V$ and $N = (U^\bullet \cap {}_\bullet V)(U_\bullet \cap {}^\bullet V)$. Let $\overline{\phi} : {}^\bullet W/{}_\bullet W \leftarrow M/N$ and $\overline{\psi} : M/N \leftarrow W^\bullet/W_\bullet$ be the isomorphisms induced by ϕ and ψ, respectively. Then $\theta_W = \overline{\phi} \circ \overline{\psi}$.*

For the sake of conciseness later, let us make a pedantic distinction. We understand a **subquotient** of R to be a group having the form M/N, constructed in the standard way, where $R \geq M \trianglerighteq N$. Note that M/N determines the pair (M, N), indeed, M is the unionset of M/N while N is the identity element. We write $[R]$ for the isomorphism class of R. We call S a **factor group** of R provided S is isomorphic to a subquotient of R. In that case, we write $[S] \leq [R]$. Thus, we impose a formal partial ordering $\leq$ on the isomorphism classes of groups.

For any $U \in \mathcal{S}(R, S)$, we define the **thorax** of U to be the group $\Theta(U)$, well-defined up to isomorphism, such that

$$^{\bullet}U / {}_{\bullet}U \cong \Theta(U) \cong U^{\bullet}/U_{\bullet} \; .$$

The latest proposition immediately yields the following three corollaries.

Corollary 2.6. *With the notation above, $[\Theta(U)] \geq [\Theta(W)] \leq [\Theta(V)]$.*

Corollary 2.7. *The morphism $U \in \mathcal{S}(R, S)$ factorizes through $\Theta(U)$. That is to say, there exist $X \in \mathcal{S}(R, \Theta(U))$ and $Y \in \mathcal{S}(\Theta(U), S)$ such that $U = X * Y$. Furthermore, U is an $\mathcal{S}$-isomorphism if and only if $\Theta(U) \cong U$.*

Corollary 2.8. *We have a group isomorphism $\mu_R : \mathrm{Aut}_{\mathcal{S}}(R) \ni \Delta(\theta) \leftarrow\!\shortmid \theta \in \mathrm{Aut}(R)$.*

We now introduce the cocycles that will appear in the definition of Λ. Let F, G, H, I be any finite groups. For brevity, we write $\ell(F) = \ell(|F|)$. Let $U \in \mathcal{S}(F, G)$, $V \in \mathcal{S}(G, H)$, $W \in \mathcal{S}(H, I)$. We define

$$\sigma(U, V) = \ell(U_{\bullet} \cap {}_{\bullet}V) \; .$$

Consider the subgroup $A = \{g{\times}h : 1{\times}g \in U, g{\times}h \in V, h{\times}1 \in W\} \leq G{\times}H$. We have

$$\frac{U_{\bullet} \cap {}_{\bullet}(V * W)}{U_{\bullet} \cap {}_{\bullet}V} \cong \frac{^{\bullet}A}{_{\bullet}A} \cong \frac{A^{\bullet}}{A_{\bullet}} \cong \frac{(U * V)_{\bullet} \cap {}_{\bullet}W}{V_{\bullet} \cap {}_{\bullet}W} \; .$$

Thus, we have given a quick proof of the equality, due to Boltje–Danz [BD13, 3.5],

$$|U_{\bullet} \cap {}_{\bullet}V|.|(U * V)_{\bullet} \cap {}_{\bullet}W| = |U_{\bullet} \cap {}_{\bullet}(V * W)|.|V_{\bullet} \cap {}_{\bullet}W| \; .$$

Since ℓ is a monoid homomorphism, the conclusion can be expressed as follows.

Proposition 2.9. (Boltje–Danz.) *With the notation above,*

$$\sigma(U, V * W)\sigma(V, W) = \sigma(U, V)\sigma(U * V, W) \; .$$

In other words, σ is a cocycle for the full subcategory of $\mathcal{S}$ on the class of finite groups.

We write $\mathbb{K}_{\sigma}\mathcal{S}$ to denote the twisted category associated with σ. To avoid misunderstanding, let us emphasize that, although the objects of $\mathcal{S}$ are arbitrary groups, the cocycle σ is defined only for finite groups and the objects of $\mathbb{K}_{\sigma}\mathcal{S}$ are arbitrary finite groups. Retaining the notation above, we write $s_U^{F,G}$ to denote U as an element of $\mathbb{K}_{\sigma}\mathcal{S}(F, G)$. The composition for $\mathbb{K}_{\sigma}\mathcal{S}(F, G)$ is given by

$$s_U^{F,G} \circ s_V^{G,H} = \sigma(U, V)\, s_{U*V}^{F,H} \; .$$

Below, we shall usually omit the $\circ$ symbol. The identity $\mathbb{K}_{\sigma}\mathcal{S}$-morphism on G is $\mathrm{id}_G = s_{\Delta(G)}^{G,G}$.

Lemma 2.10. *Let E, F, G be finite groups and $U \in \mathcal{S}(F, G)$ such that $E \cong \Theta(U)$. Then there exist $X \in \mathcal{S}(F, E)$ and $Y \in \mathcal{S}(E, G)$ satisfying $U = X * Y$. For any such X and Y, we have $s_U^{F,G} = s_X^{F,E} s_Y^{E,G}$.*

Proof. Corollary 2.7 implies the existence of X and Y. Corollary 2.6 implies that $\Theta(X) \cong E$. Hence, $X_{\bullet} = 1$ and $\sigma(X, Y) = 1$. □

3 Classification of simple modules

To apply ring theoretic techniques, we replace $\mathbb{K}_\sigma\mathcal{S}$ with the small full subcategory

$$\Lambda_\mathcal{K} = \mathbb{K}_\sigma\mathcal{S}_\mathcal{K} \, .$$

Although $\Lambda_\mathcal{K}$ is determined by the triple $(\mathcal{K}, \mathbb{K}, \ell)$, we omit $\mathbb{K}$ and ℓ from the notation because we shall always be treating them as fixed. Let us point out that the dependence on ℓ is a dependence only on the values $\ell(q)$ where q runs over the prime divisors of the orders of the elements of $\mathcal{K}$. When no ambiguity can arise, we write $\Lambda = \Lambda_\mathcal{K}$. Employing an abuse of notation discussed in the previous section, the algebra associated with the category Λ will also be written as Λ. We shall be making a study of the algebra Λ, which we view as coming equipped with the complete family of mutually orthogonal idempotents $(\mathrm{id}_G^\Lambda : G \in \mathcal{K})$. We can also view Λ as the twisted category algebra associated with the restriction of σ to $\mathcal{K}$. In this section, we shall be making much use of the $\mathbb{K}$-basis $\{s_U^{F,G} : F \in \mathcal{K} \ni G, U \in \mathcal{S}(F,G)\}$ for Λ. We call that basis the **square basis**.

Let $F, G, H \in \mathcal{K}$. Given group isomorphisms $\phi : F \leftarrow G$ and $\psi : G \leftarrow H$, then

$$s_{\Delta(\phi)}^{F,G} \, s_{\Delta(\psi)}^{G,H} = s_{\Delta(\phi\psi)}^{F,H} \, .$$

So, specializing and reinterpreting the map in Corollary 2.8, we have a group monomorphism

$$\mu_G \, : \, \mathrm{Aut}_\Lambda(G) \ni s_{\Delta(\theta)}^{G,G} \leftarrow\!\shortmid \theta \in \mathrm{Aut}(G) \, .$$

Extending $\mathbb{K}$-linearly, we obtain a unity-preserving algebra monomorphism

$$\mu_G \, : \, \mathrm{End}_\Lambda(G) \leftarrow \mathbb{K}\mathrm{Aut}(G) \, .$$

By Corollary 2.7, the set of non-isomorphisms in the monoid $\mathrm{End}_\mathcal{S}(G) = \mathcal{S}(G,G)$ is

$$\mathcal{S}(G,G)_< = \{U \in \mathcal{S}(G,G) : [\Theta(U)] < [G]\} \, .$$

Let $\mathrm{End}_\Lambda(G)_<$ denote the $\mathbb{K}$-submodule of $\mathrm{End}_\Lambda(G)$ with $\mathbb{K}$-basis $\{s_U^{G,G} : U \in \mathcal{S}(G,G)_<\}$. By Corollary 2.6, $\mathrm{End}_\Lambda(G)_<$ is an ideal of $\mathrm{End}_\Lambda(G)$. The next result follows.

Proposition 3.1. *Let $G \in \mathcal{K}$. Then*

$$\mathrm{End}_\Lambda(G) = \mu_G(\mathbb{K}\mathrm{Aut}(G)) \oplus \mathrm{End}_\Lambda(G)_<$$

as the direct sum of a common-unity subalgebra and an ideal.

Digressing to introduce some general notation, let A and B be rings and $\theta : A \leftarrow B$ a homomorphism. We write ${}_A\mathrm{Ind}_B^\theta$ for induction to A-modules from B-modules via θ. We write ${}_B\mathrm{Res}_A^\theta$ for restriction via θ. When θ is an isomorphism, we write ${}_A\mathrm{Iso}_B^\theta = {}_A\mathrm{Ind}_B^\theta = {}_A\mathrm{Res}_B^{\theta^{-1}}$. We sometimes omit the subscripts A and B.

We define an **$\mathcal{S}$-seed** for $\mathcal{K}$ to be a pair (E, W) such that E is a factor group of an element of $\mathcal{K}$ and W is a simple $\mathbb{K}\mathrm{Aut}(E)$-module. Two such pairs (E, W) and (E', W') are said to be **equivalent** provided there exists a group isomorphism $\theta : E \leftarrow E'$ such that $W \cong \mathrm{Iso}^\theta(V)$. In Theorems 3.4 and 3.5, for arbitrary $\mathcal{K}$, we shall be describing a bijective correspondence between the isomorphism classes of simple Λ-modules and the equivalence classes of $\mathcal{S}$-seeds for $\mathcal{K}$. The next theorem, too, describes such a correspondence, but under a strong hypothesis on $\mathcal{K}$. For other group categories, analogous theorems, with similar hypotheses on the set of objects, can be found in, for instance, Thévenaz–Webb [TW95, Section 2], Bouc [Bou10, 4.3.10]. We say that $\mathcal{K}$ is **closed under factor groups up to isomorphism** provided $\mathcal{K}$ owns an isomorphic copy of every subquotient of every element of $\mathcal{K}$. Note, this is a strong and sometimes inconvenient condition which excludes the important case where $\mathcal{K}$ consists of a single non-trivial finite group.

Theorem 3.2. *Suppose $\mathcal{K}$ is closed under factor groups up to isomorphism. Then every $\mathcal{S}$-seed for $\mathcal{K}$ is equivalent to an $\mathcal{S}$-seed (E,W) for $\mathcal{K}$ such that $E \in \mathcal{K}$. The isomorphism classes $[S]$ of simple Λ-modules S and the equivalence classes $[E,W]$ of $\mathcal{S}$-seeds (E,W) for $\mathcal{K}$ are in a bijective correspondence $[S] \leftrightarrow [E,W]$ characterized by the condition that, if $E \in \mathcal{K}$, then, with respect to the partial ordering of isomorphism classes of groups, $[E]$ is minimal subject to $S(E) \neq 0$ and $W \cong \mathrm{Res}^{\mu_E}(S(E))$.*

Proof. The first sentence of the conclusion is obvious. By Lemma 2.10 and the hypothesis on $\mathcal{K}$, every Λ-morphism $F \leftarrow G$ factorizes through a group $T \in \mathcal{K}$ such that $[F] \geq [T] \leq [G]$. Perforce, in the terminology of [BD16], the $\mathbb{K}$-linear category Λ is admissible with respect to the above partial ordering. The required conclusion now follows from [BD16, 2.4]. □

To generalize the correspondence in the latest theorem, we need a lemma.

Lemma 3.3. *Suppose $\mathcal{K}$ is closed under subquotients up to isomorphism. Let $[S] \leftrightarrow [E,W]$ in the notation of the latest theorem. Let $G \in \mathcal{K}$. Then $S(G) \neq 0$ if and only if E is a factor group of G.*

Proof. In one direction, the conclusion is already part of the characterization of S. Conversely, suppose G has a subquotient B/Y isomorphic to E. Replacing E with an isomorphic copy, we may assume that $E \in \mathcal{K}$. Let $x \in S(E) - \{0\}$. Let $\phi : E/1 \leftarrow B/Y$ be an isomorphism. Let $I = \Delta(E,1,\phi,Y,B)$ and $J = \Delta(B,Y,\phi^{-1},1,E)$. Then $I * J = \Delta(E)$ and $\sigma(I,J) = \ell(Y)$, so

$$s_I^{E,G} s_J^{G,E} x = \ell(Y) s_{\Delta(E)}^{E,E} x = \ell(Y) x \neq 0 \, .$$

So the element $s_J^{G,E} x \in S(G)$ is non-zero. □

Now let $\mathcal{K}$ be arbitrary. Given an $\mathcal{S}$-seed (E,W) for $\mathcal{K}$, we define $S_{E,W}$ to be the simple Λ-module, well-defined up to isomorphism, defined as follows. Choose any set $\mathcal{K}'$ of finite groups such that $\mathcal{K} \subseteq \mathcal{K}'$ and $\mathcal{K}'$ is closed under subquotients up to isomorphism. Let $\Lambda' = \Lambda_{\mathcal{K}'}$. Since (E,W) is an $\mathcal{S}$-seed for $\mathcal{K}'$, there exists an isomorphically unique simple Λ'-module S' corresponding to (E,W) as in Theorem 3.2. We define $S_{E,W} = \Lambda S'$, which is a simple Λ-module by Proposition 2.1 and Lemma 3.3. To check that $S_{E,W}$ is independent of the choice of $\mathcal{K}'$, let $\mathcal{K}''$ be another such set of finite groups. Write $\mathcal{K}''' = \mathcal{K}' \cup \mathcal{K}''$. Let S'' and S''' be the simple modules defined in the same way as S' but with $\mathcal{K}''$ and $\mathcal{K}'''$, respectively, in place of $\mathcal{K}'$. By considering a diamond diagram of corner embeddings, we deduce that $\Lambda S' \cong \Lambda S''' \cong \Lambda S''$. The well-definedness is now established. We say that $S_{E,W}$ has **$\mathcal{S}$-seed** (E,W) and **minimal group** E. We say that (E,W) and E are **associated** with $S_{E,W}$.

Let us point out that the Λ-modules $\Lambda S'$, $\Lambda S''$, $\Lambda S'''$ can be viewed as restrictions. Indeed, in the scenario of Proposition 2.1, we can make an identification $BN = {}_B\mathrm{Res}_A(N)$ for any A-module N. The next remark consolidates an observation that is already implicit above.

Remark 3.4. *Let (E, W) be an $\mathcal{S}$-seed for $\mathcal{K}$. Let $\mathcal{L} \subseteq \mathcal{K}$ such that E is a factor group of a group in $\mathcal{L}$. Then (E, W) is an $\mathcal{S}$-seed for $\mathcal{L}$ and the restricted $\Lambda_{\mathcal{L}}$-module $\Lambda_{\mathcal{L}}.S_{E,W}$ has $\mathcal{S}$-seed (E, W).*

Let us emphasize that, in the next result, $\mathcal{K}$ is arbitrary. In particular, we do not require $\mathcal{K}$ to own isomorphic copies the minimal groups associated with the simple Λ-modules. This is in contrast to the situation for some other kinds of group functors, where work without such an ownership condition tends to be very difficult. Several examples in Bouc–Stancu–Thévenaz [BST13, Section 13] show that, for biset functors, there are no direct analogues of the latest lemma or the two theorems below in this section. See also the discussion of the "vanishing problem" in Rognerud [Rog19].

Theorem 3.5. *The condition $S \cong S_{E,W}$ characterizes a bijective correspondence $[S] \leftrightarrow [E, W]$ between the isomorphism classes $[S]$ of simple Λ-modules S and the equivalence classes $[E, W]$ of $\mathcal{S}$-seeds (E, W) for $\mathcal{K}$.*

Proof. Let $\mathcal{K}'$ and Λ' be as defined just above. Theorem 3.2 describes the simple Λ'-modules. The argument is completed by applying Proposition 2.1 to the corner embedding $\Lambda' \hookleftarrow \Lambda$ and then making use of Lemma 3.3 and Remark 3.4. □

The next theorem describes the same correspondence in a more intrinsic way, without mentioning any extension of the set of objects. Let E be a finite group, let $\mathcal{K} \ni G \geq B \trianglerighteq Y$ and let $\phi : E \leftarrow B/Y$ be an isomorphism. We write

$$\mu_\phi \, : \, \mathrm{End}_\Lambda(G) \leftarrow \mathbb{K}\mathrm{Aut}(E)$$

for the algebra monomorphism such that, given $\epsilon \in \mathrm{Aut}(E)$, then

$$\mu_\phi(\epsilon) = \ell(Y)^{-1} s^{G,G}_{\Delta(B,Y,\phi^{-1}\epsilon\phi,Y,B)} \, .$$

Theorem 3.6. *The correspondence in the previous theorem is characterized as follows. Given a simple Λ-module S then, up to equivalence, the $\mathcal{S}$-seed (E, W) for $\mathcal{K}$ associated with S is determined by the following three conditions:*
(a) *For any $G \in \mathcal{K}$, we have $S(G) \neq 0$ if and only if $[E] \leq [G]$.*
(b) *Given $\mathcal{K} \ni G \geq B \trianglerighteq Y$, an isomorphism $\phi : E \leftarrow B/Y$ and an idempotent k of $\mathbb{K}\mathrm{Aut}(E)$, then $\mu_\phi(k)S(G) \neq 0$ if and only if $kW \neq 0$.*
(c) *The isomorphism class $[E]$ is minimal in the sense that, given an $\mathcal{S}$-seed (E', W') for $\mathcal{K}$ satisfying conditions (a) and (b), then $[E] \leq [E']$.*

Proof. By Proposition 2.1 and Remark 3.4, we may assume that S has $\mathcal{S}$-seed (E, W) and that $E \in \mathcal{K}$. It suffices to deduce conditions (a), (b), (c). Lemma 3.3 yields condition (a) which, together with Theorem 3.2, implies condition (c). Let $I = \Delta(E, 1, \phi, Y, B)$ and $J = \Delta(B, Y, \phi^{-1}, E, 1)$. Given $\epsilon \in \mathrm{Aut}(E)$, then

$$s^{E,G}_I \mu_\phi(\epsilon) s^{G,E}_J = \ell(Y) \mu_E(\epsilon) \, .$$

By $\mathbb{K}$-linearity, the same equality holds with k in place of ϵ. We have $kW \neq 0$ if and only if $\mu_E(k)S(E) \neq 0$. In that case, $\mu_\phi(\epsilon) s^{G,E}_J S(E) \neq 0$, hence $\mu_\phi(\epsilon)S(G) \neq 0$. Conversely, suppose $S(G) \neq 0$. Noting that

$$s^{G,E}_J \mu_E(\epsilon) s^{E,G}_I = \ell(Y) \mu_\phi(\epsilon) \, .$$

and arguing as before, we obtain $kW \neq 0$. We have deduced condition (b). □

Let us mention that, if we were to generalize by allowing ℓ to be any homomorphism of multiplicative monoids $\mathbb{K}^\times \leftarrow \mathbb{N} - \{0\}$ then, adopting a suitable extension of the notion of a twisted category algebra, the discussion in this section would carry through, up to and including Theorem 3.2, but the proof of Lemma 3.3 would no longer be valid. We do not know of any examples for which the conclusion of that lemma would fail.

Recall, for a ring A, a primitive idempotent of the centre $Z(A)$ is called a **block** of A. When A is unital and semisimple, the blocks of A are in a bijective correspondence with the simple A-modules up to isomorphism, each block corresponding to the isomorphically unique simple module of the associated block algebra. If $\mathcal{K}$ is finite then, for each $\mathcal{S}$-seed (E, W) for $\mathcal{K}$, we let $b_{E,W}$ denote the block of Λ that acts as the identity on $S_{E,W}$.

Example 3.7. *Let q be a prime and $\mathcal{K} = \{G\}$ with $G \cong C_q$, the cyclic group with order q. Suppose $\mathbb{K}$ has a root of unity with order $q-1$. Then the algebra $\Lambda = \mathrm{End}_\Lambda(G)$ has a $\mathbb{K}$-basis consisting of the elements*

$$s_0 = s^{G,G}_{1\times 1}\,, \quad s_{01} = s^{G,G}_{1\times G}\,, \quad s_{10} = s^{G,G}_{G\times 1}\,, \quad s_{11} = s^{G,G}_{G\times G}\,, \quad s_d = s^{G,G}_{\Delta(d)}$$

where $\Delta(d) = \{(g^d, g) : g \in G\}$ and d runs over the elements of the unit group $(\mathbb{Z}/q)^\times$ of the ring $\mathbb{Z}/q$ of modulo q congruence classes. The $\mathcal{S}$-seeds for $\mathcal{K}$ are $(1,1)$, (G,ζ), (G,χ) where ζ is a trivial $\mathbb{K}\mathrm{Aut}(G)$-module and χ runs over representatives of the $q-2$ isomorphism classes of non-trivial simple $\mathbb{K}\mathrm{Aut}(G)$-modules, vacuously when $q = 2$. Write $\lambda = \ell(q)$ and $r = \lambda s_0 - s_{01} - s_{10} + s_{11}$.
(1) *If $\lambda = 1$, then $\mathbb{K}r$ is a nilpotent ideal of Λ. In particular, Λ is not semisimple.*
(2) *Suppose $\lambda \neq 1$. Then Λ is semisimple. Identifying each χ with the associated irreducible character, the blocks of Λ are*

$$b_{1,1} = \frac{r}{\lambda - 1}\,, \qquad b_{G,\zeta} = \frac{-r}{\lambda - 1} + \frac{1}{q-1}\sum_d s_d\,, \qquad b_{G,\chi} = \frac{1}{q-1}\sum_d \chi(d^{-1}) s_d\,.$$

We have $\Lambda b_{1,1} \cong \mathrm{Mat}_2(\mathbb{K})$ and $\Lambda b_{G,\zeta} \cong \Lambda b_{G,\chi} \cong \mathbb{K}$.

Proof. This exercise in laborious but straightforward calculation can be done in many different ways. Let us sketch a fairly quick route. Suppose $\lambda \neq 1$. It is easy to check that the regular Λ-module has a submodule S with $\mathbb{K}$-basis $\{s_0, s_{10}\}$ and representation given, with respect to that basis, by

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow s_0\,, \quad \begin{bmatrix} 1 & \lambda \\ 0 & 0 \end{bmatrix} \leftarrow s_{01}\,, \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \leftarrow s_{10}\,, \quad \begin{bmatrix} 0 & 0 \\ 1 & \lambda \end{bmatrix} \leftarrow s_{11}\,, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow s_d\,.$$

Plainly, S is simple and $\mathrm{End}_\Lambda(S) \cong \mathbb{K}$. By Theorem 3.5, Λ has exactly q simple modules up to isomorphism. By counting dimensions, Λ is semisimple. Using the above matrices, it is easy to show that $Z(\Lambda)$ has a basis consisting of r and the elements s_d. We have $b_{1,1} \in Z(\Lambda) \cap \Lambda_< = \mathbb{K}r$. The formula for $b_{1,1}$ follows easily.

Let ψ run over all $q-1$ of the irreducible characters of $\mathbb{K}\mathrm{Aut}(G)$. Define $b_\psi = \sum_d \psi(d^{-1}) s_d$. Direct calculations show that $\sum_\psi b_\psi = 1$ as a sum of mutually orthogonal idempotents of $Z(\Lambda)$. By considering actions on simple $\mathbb{K}\mathrm{Aut}(G)$-modules and noting that $b_{1,1} b_\zeta = b_{1,1}$, we deduce that $b_{1,1} + b_{G,\zeta} = b_\zeta$ and each $b_{G,\chi} = b_\chi$. □

Generally, we can identify Λ with $\mathbb{K}\mathcal{S}_\mathcal{K}$ as $\mathbb{K}$-modules by identifying each $s^{F,G}_U$ with the element $U \in \mathcal{S}(F, G)$. In fact, that identification of underlying $\mathbb{K}$-modules is already implicit in our construction of Λ. The example shows that the isomorphism in Theorem 1.2 can depend on ℓ. Indeed, in the notation of the example, putting $\lambda \neq 1$ then, as $\mathbb{K}$-submodules of $\mathbb{K}\mathcal{S}_{\mathcal{K}}$, the block algebras $\Lambda b_{1,1}$ and $\Lambda b_{G,\chi}$ are independent of ℓ, but the block algebra $\Lambda b_{G,\zeta}$ does depend on ℓ. Moreover, again as $\mathbb{K}$-submodules of $\mathbb{K}\mathcal{S}_{\mathcal{K}}$, the first summand on the right-hand side of the algebra isomorphism

$$\Lambda \cong \mathrm{Mat}_2(\mathbb{K}\mathrm{Aut}(1)) \oplus \mathrm{Mat}_1(\mathbb{K}\mathrm{Aut}(G))$$

is constant, but the other summand varies.

4 Implications of semisimplicity

In this section, we prove Theorem 1.2.

For convenience, we constrain the notion of a Λ-module by deeming that, for any Λ-module M, the $\mathbb{K}$-module $M(G)$ is finite-dimensional for all $G \in \mathcal{K}$, equivalently, eM is finite-dimensional for all idempotents e of Λ. The condition ensures that every simple factor of M has a well-defined finite multiplicity. Given $G \in \mathcal{K}$, then the module $M(G)$ of $\mathrm{End}_\Lambda(G)$ has submodule $\mathrm{End}_\Lambda(G)_<.M(G)$ and we can define a $\mathbb{K}\mathrm{Aut}(G)$-module

$$\overline{M}(G) = \mathrm{Res}^{\mu_G}(M(G)/\mathrm{End}_\Lambda(G)_<.M(G)) \,.$$

Given an $\mathcal{S}$-seed (E,W) for $\mathcal{K}$, we let $m_{E,W}(M)$ denote the multiplicity of $S_{E,W}$ as a simple factor of M. When Λ is locally semisimple,

$$M \cong \bigoplus m_{E,W}(M)\, S_{E,W}$$

summed over representatives (E,W) of the equivalence classes of $\mathcal{S}$-seeds for $\mathcal{K}$.

Lemma 4.1. *Suppose Λ is locally semisimple. Let (E,W) be an $\mathcal{S}$-seed for $\mathcal{K}$ such that $E \in \mathcal{K}$. Let M be a Λ-module. Then $m_{E,W}(M)$ is the multiplicity of W in $\overline{M}(E)$.*

Proof. Write $\mathcal{E} = \mathrm{End}_\Lambda(E)$. We may assume that M is simple. Let (E',W') be an $\mathcal{S}$-seed for $\mathcal{K}$ associated with M. If $\overline{M}(E) = 0$, then $[E',W'] \neq [E,W]$ and $m_{E,W} = 0$. So we may assume that $\overline{M}(E) \neq 0$. By condition (c) of Theorem 3.6, $[E'] \leq [E]$. Let $E \geq B \trianglerighteq Y$ and let $\phi : E' \leftarrow B/Y$ be an isomorphism. By condition (b) of the same theorem, $\mu_\phi(\epsilon')M(E) \neq 0$ for some $\epsilon' \in \mathrm{Aut}(E')$. For a contradiction, suppose $[E'] < [E]$. The proof of the theorem shows that $\mu_\phi(\epsilon') \in \mathcal{E}_<$. Therefore, $\mathcal{E}_<.M(E) \neq 0$. But, applying Proposition 2.1 to the corner subalgebra $\mathcal{E}$ of Λ, we deduce that $M(E)$ is a simple $\mathcal{E}$-module. Therefore $\mathcal{E}_<.M(E) = M(E)$, contradicting an assumption on M. We have shown that M has minimal group E. That reduces to the case $\mathcal{K} = \{E\}$, for which the required conclusion is clear. □

Now letting (E,W) be any $\mathcal{S}$-seed for $\mathcal{K}$, we let m_W denote the multiplicity of W in the regular $\mathbb{K}\mathrm{Aut}(E)$-module. Of course, when $\mathbb{K}$ is algebraically closed, $m_W = \dim_{\mathbb{K}}(W)$. For $G \in \mathcal{K}$, we let n_E^G denote the number of subquotients of G isomorphic to E.

Lemma 4.2. *Suppose Λ is locally semisimple. Given $G \in \mathcal{K}$ and an $\mathcal{S}$-seed (E,W) for $\mathcal{K}$, then $m_{E,W}(\Lambda.\mathrm{id}_G^\Lambda) = n_E^G m_W$.*

Proof. Let $\mathcal{K}' = \mathcal{K} \cup \{E\}$ and $\Lambda' = \Lambda_{\mathcal{K}'}$. Noting that $\mathrm{id}_G^{\Lambda'} = s_{\Delta(G)}^{G,G} = \mathrm{id}_G^{\Lambda}$, Remark 3.4 allows us to replace $\mathcal{K}$ with $\mathcal{K}'$. So we may assume that $E \in \mathcal{K}$. Write $M = \Lambda.\mathrm{id}_G^{\Lambda}$ and, again, $\mathcal{E} = \mathrm{End}_\Lambda(E)$. Then $M = \bigoplus_{F \in \mathcal{K}} \Lambda(F,G)$ and $M(E) = \Lambda(E,G)$. Given $U \in \mathcal{S}(E,G)$ and writing $U = \Delta(A,X,\phi,Y,B)$, then

$$s_{\Delta(A,X,\mathrm{id},X,A)}^{E,E} s_U^{E,G} = \ell(X) s_U^{E,G}$$

which belongs to $\mathcal{E}_<.M(E)$ unless $A = E$ and $X = 1$. So

$$\overline{M}(E) = \bigoplus_{\phi, B/Y} \mathbb{K}(s_{\Delta(E,1,\phi,Y,B)}^{E,G} + \mathcal{E}_<.M(E))$$

as $\mathbb{K}$-modules, where B/Y runs over the subquotients of G isomorphic to E and ϕ runs over the isomorphisms $E \leftarrow B/Y$. Given $\epsilon \in \mathrm{Aut}(E)$, then

$$\mu_E(\epsilon) s_{\Delta(E,1,\phi,Y,B)}^{E,G} = s_{\Delta(E,1,\epsilon\circ\phi,Y,B)}^{E,G} .$$

So $\overline{M}(E)$ is isomorphic to the direct sum of n_E^G copies of the regular $\mathbb{K}\mathrm{Aut}(E)$-module. An appeal to Lemma 4.1 completes the argument. □

Lemma 4.3. *Given an $\mathcal{S}$-seed (E,W) for $\mathcal{K}$, then* $\mathrm{End}_\Lambda(S_{E,W}) \cong \mathrm{End}_{\mathbb{K}\mathrm{Aut}(E)}(W)$ *as an isomorphism of division algebras over* $\mathbb{K}$.

Proof. The ring isomorphisms in the proof of Lemma 2.2 are $\mathbb{K}$-linear when, in the notation of that lemma, A is an algebra over $\mathbb{K}$ and B is a subalgebra. □

Given an $\mathcal{S}$-seed (E,W) for $\mathcal{K}$, we let Δ_W denote the opposite ring of $\mathrm{End}_{\mathbb{K}\mathrm{Aut}(E)}(W)$. Up to isomorphism, Δ_E is determined by the condition that

$$\mathbb{K}\mathrm{Aut}(E) b_W \cong \mathrm{Mat}_{m_W}(\Delta_W)$$

where b_W is the block of $\mathbb{K}\mathrm{Aut}(E)$ fixing W.

Theorem 4.4. *Suppose $\mathcal{K}$ is finite and Λ is semisimple. Given an $\mathcal{S}$-seed (E,W) for $\mathcal{K}$, then*

$$\Lambda b_{E,W} \cong \mathrm{Mat}_{n_E m_W}(\Delta_{E,W})$$

where $\Delta_{E,W}$ is the opposite algebra of $\mathrm{End}_{\mathbb{K}\mathrm{Aut}(E)}(W)$.

Proof. The regular Λ-module decomposes as ${}_\Lambda\Lambda = \bigoplus_{G \in \mathcal{K}} \Lambda.\mathrm{id}_G^{\Lambda}$. Since $n_E = \sum_G n_E^G$, Lemma 4.2 gives $m_{E,W}({}_\Lambda\Lambda) = n_E m_W$. So

$${}_\Lambda\Lambda b_{E,W} \cong n_E m_W S_{E,W}$$

as Λ-modules. To complete the argument, we apply Lemma 4.3. □

Theorem 1.2 follows because $\Lambda = \bigoplus_{E,W} \Lambda b_{E,W}$, summed over the $\mathcal{S}$-seeds (E,W) up to equivalence.

5 The round basis

After Boltje–Danz [BD13], we introduce a basis for Λ, called the **round basis**. We review their characterization [BD13, Sections 2, 3] of the product of two elements of that basis. Our reformulation, in terms of changes of coodinates rather than changes of associative operation, will suit our applications in later sections.

We refer to Stanley [Sta11, Chapter 3] for some terminology pertaining to posets. Let $\mathcal{P}$ be a poset such that every finite subset of $\mathcal{P}$ generates a finite order ideal, equivalently, for all $u \in \mathcal{P}$, the principal order ideal $(-,u]_{\mathcal{P}} = \{v \in \mathcal{P} : v \leq u\}$ is finite. See [Sta11, 3.7, 3.8] for an introduction to the theory of the Möbius function $\text{möb}_{\mathcal{P}} : \mathbb{Z} \leftarrow \mathcal{P}{\times}\mathcal{P}$. A well-known and straightforward generalization of [Sta11, 3.7.1] asserts that, given an abelian group $\mathcal{A}$ and funtions $\sigma, \tau : \mathcal{A} \leftarrow \mathcal{P}$, then the following two conditions are equivalent:

- we have $\sigma(u) = \sum_{v \in (-,u]_{\mathcal{P}}} \tau(v)$ for all $u \in \mathcal{P}$,
- we have $\tau(u) = \sum_{v \in \mathcal{P}} \sigma(v)\text{möb}(v,u)$ for all $v \in \mathcal{P}$.

We call σ the **sum function** of τ. We call τ the **totient function** of σ. Let us mention that the terminology reflects the intended sense in the origin of the word *totient*, Sylvester [Sy888].

A **downward retraction** of $\mathcal{P}$ is defined to be a decreasing idempotent endomorphism of $\mathcal{P}$, we mean, a function $\rho : \mathcal{P} \leftarrow \mathcal{P}$ such that $\rho^2(u) = \rho(u) \leq u$ for all $u \in \mathcal{P}$. All the retractions we consider will be downward retractions. The next result is in Boltje–Danz [BD13, 4.1]. Let us give a quick alternative proof.

Lemma 5.1. (Boltje–Danz.) *Let $\mathcal{P}$ and $\mathcal{A}$ be as above, ρ a retraction of $\mathcal{P}$ and $\sigma : \mathcal{A} \leftarrow \mathcal{P}$ a function such that $\sigma(u) = \sigma(\rho(u))$ for all $u \in \mathcal{P}$. Write σ' for the restriction of σ to the subposet $\mathcal{P}' = \rho(\mathcal{P})$. Let τ and τ' be the totient functions of σ and σ' on $\mathcal{P}$ and $\mathcal{P}'$, respectively. Then $\tau(v) = \tau'(v)$ for all $v \in \mathcal{P}'$ and $\tau(v) = 0$ for all $v \in \mathcal{P} - \mathcal{P}'$.*

Proof. Let $v \in \mathcal{P}$. First consider the case $v \notin \mathcal{P}'$. Writing $v' = \rho(v)$, then

$$0 = \sigma(v) - \sigma(v') = \tau(v) + \sum\nolimits_w \tau(w)$$

where w runs over those elements of $\mathcal{P}$ such that $v' \ngeq w < v$. Since ρ is an endomorphism, each $\rho(w) \leq v' \neq w$, hence $w \notin \mathcal{P}'$. By an inductive argument on the height of v, we may assume that each $\tau(w) = 0$, hence $\tau(v) = 0$. Now consider the case $v \in \mathcal{P}'$. By the conclusion for the other case,

$$\tau(v) + \sum_{w \in (-,v)_{\mathcal{P}'}} \tau(w) = \sum_{w \in (-,v]_{\mathcal{P}}} \tau(w) = \sigma(v) = \sigma'(v) = \tau'(v) + \sum_{w \in (-,v)_{\mathcal{P}'}} \tau'(w)$$

where the notation indicates that two of the summations are over open intervals. Another inductive argument on height yields $\tau(v) = \tau'(v)$. □

Let F, G, H be finite groups. We write $\text{möb}(F,G)$ for the value, at (F,G), of the Möbius function of the formal poset of finite groups, partially ordered by inclusion. Let $I \in \mathcal{S}(F,G)$, $J \in \mathcal{S}(G,H)$, $K \in \mathcal{S}(F,H)$. We define

$$\mathcal{P}_K^{I,J} = \{(U,V) \in \mathcal{S}(F,G) \times \mathcal{S}(G,H) : K \leq U * V, (U,V) \leq (I,J)\}$$

as a subposet of the direct product $\mathcal{S}(F,G){\times}\mathcal{S}(G,H)$. We define

$$\tau_K^{I,J} = \sum_{(U,V) \in \mathcal{P}_K^{I,J}} \text{möb}(U,I)\text{möb}(V,J)\sigma(U,V) \, .$$

We call the pair (I,J) **strongly compatible** provided $I^\bullet = {}^\bullet J$. Note, the condition implies that ${}^\bullet(I*J) = {}^\bullet I$ and $(I*J)^\bullet = J^\bullet$. Given W such that $K \leq W \in \mathcal{S}(F,H)$, we call K an **adequate subgroup** of W provided ${}^\bullet K = {}^\bullet W$ and $K^\bullet = W^\bullet$. Let $\mathrm{ad}(W)$ denote the set of adequate subgroups of W. We point out that, if (I,J) is strongly compatible and $K \in \mathrm{ad}(I*J)$, then ${}^\bullet K = {}^\bullet I$ and $K^\bullet = J^\bullet$.

We define $\{t_I^{F,G} : I \in \mathcal{S}(F,G)\}$ to be the $\mathbb{K}$-basis for $\mathbb{K}_\sigma\mathcal{S}(F,G)$ given by the two equivalent equalities

$$s_U^{F,G} = \sum_{I \in \mathcal{S}(U)} t_I^{F,G} , \qquad t_I^{F,G} = \sum_{U \in \mathcal{S}(I)} \text{möb}(U,I)\, s_U^{F,G}$$

where the first equality holds for all $U \in \mathcal{S}(F,G)$, the second, for all $I \in \mathcal{S}(F,G)$. To confirm the equivalence, observe that the functions $\mathbb{K}_\sigma\mathcal{S}(F,G) \leftarrow \mathcal{S}(F{\times}G)$ given by $s_U^{F,G} \leftarrow\!\shortmid U$ and $t_I^{F,G} \leftarrow\!\shortmid I$ are, respectively, the sum function and the totient function of each other. We point out that the basis $\{t_I^{F,G} : I\}$ of $\mathbb{K}_\sigma\mathcal{S}(F,G)$ can be regarded simply as a basis of the $\mathbb{K}$-module $\mathbb{K}\mathcal{S}(F,G)$, well-defined independently of ℓ. However, we shall be concerned with composites of the basis elements in $\mathbb{K}_\sigma\mathcal{S}$. The composites do depend on ℓ.

The set $\{t_I^{F,G} : F \in \mathcal{K} \ni G, I \in \mathcal{S}(F,G)\}$ is another $\mathbb{K}$-basis for Λ. We call it the **round basis**. Let us mention that the round basis has a long history, a version of it going back to the 19th century, implicitly appearing in Burnside's celebrated table of marks. We shall be working with the round basis of Λ in later sections. For now, though, there is no need to pass down to a small subcategory of $\mathbb{K}_\sigma\mathcal{S}$.

The next result is [BD13, 3.9, 4.2, 4.3]. In some of our applications of the formulas for $\tau_K^{I,J}$, we shall be considering the poset retraction that appears in the proof.

Theorem 5.2. (Boltje–Danz.) *Let F, G, H be finite groups. Let $I \in \mathcal{S}(F,G)$ and $J \in \mathcal{S}(G,H)$. Then:*

(1) *We have*

$$t_I^{F,G} t_J^{F,G} = \sum_{K \in \mathcal{S}(F,H)} \tau_K^{I,J} t_K^{I,J} .$$

(2) *For any $K \in \mathcal{S}(F,H)$, if $\tau_K^{I,J} \neq 0$, then (I,J) is strongly compatible and $K \in \mathrm{ad}(I*J)$.*
(3) *If $t_I^{F,G} t_J^{G,H} \neq 0$, then (I,J) is strongly compatible.*
(4) *If (I,J) is strongly compatible then, for all $K \in \mathrm{ad}(I*J)$, we have*

$$\tau_K^{I,J} = \sum_{(U,V) \in \mathcal{R}_K^{I,J}} \text{möb}_{\mathcal{R}_K^{I,J}}((U,V),(I,J))\sigma(U,V)$$

where $\mathcal{R}_K^{I,J} = \{(U,V) \in \mathcal{P}_K^{I,J} : U^\bullet = {}^\bullet V\}$.

Proof. A straightforward manipulation yields part (1). Note that, if $K \nleq I*J$, then $\mathcal{P}_K^{I,J} = \varnothing$ and $\tau_K^{I,J} = 0$. Now fix $K \leq I*J$. By the convexity of $\mathcal{P}_K^{I,J}$ in $\mathcal{S}(F,G){\times}\mathcal{S}(G,H)$ together with the formula in [Sta11, 3.8.2] for the Möbius function of a direct product of posets,

$$\text{möb}(U,I)\text{möb}(V,J) = \text{möb}_{\mathcal{P}_K^{I,J}}((U,V),(I,J))$$

for each $(U,V) \in \mathcal{P}_K^{I,J}$. Let

$$\Gamma_K(U,V) = \{f{\times}g{\times}h \in F{\times}G{\times}H : f{\times}g \in U, f{\times}h \in K, g{\times}h \in V\} .$$

Let $S_{U,V} = \{f{\times}g : f{\times}g{\times}h \in \Gamma_K(U,V)\}$ and $T_{U,V} = \{g{\times}h : f{\times}g{\times}h \in \Gamma_K(U,V)\}$, which are subgroups of U and V, respectively. The function

$$\rho_K^{I,J} : \mathcal{P}_K^{I,J} \ni (S_{U,V}, T_{U,V}) \leftarrow\!\shortmid (U,V) \in \mathcal{P}_K^{I,J}$$

is easily shown to be a retraction that preserves σ and has image $\mathcal{R}_K^{I,J}$. Parts (2), (3), (4) now follow from Lemma 5.1. □

6 Semisimplicity and endomorphism algebras

Rognerud [Rog19, 7.3] gave a quick proof of Bouc's special case of Theorem 1.4 by showing that the given category is locally semisimple if and only if the endomorphism algebras of all the objects are semisimple. In this section, we shall pursue a similar theme, though the results are not directly analogous.

The following observations, employed only incidentally in the next proof, will be used more substantially in the next section. Let R, S, T be groups and $U \in \mathcal{S}(R,S)$ and $V \in \mathcal{S}(S,T)$. The **opposite** of U is defined to be the morphism $U^\circ \in \mathcal{S}(S,R)$ such that $U^\circ = \{s{\times}r : r{\times}s \in U\}$. Plainly, $(U{*}V)^\circ = V^\circ{*}U^\circ$. Now let F, G, H be finite groups. Let $U \in \mathcal{S}(F,G)$ and $V \in \mathcal{S}(G,H)$. Then $\sigma(U,V) = \sigma(V^\circ, U^\circ)$. So, now taking F, G, H to be elements of $\mathcal{K}$, there is a self-inverse antiautomorphism $\Lambda \ni u^\circ \leftrightarrow u \in \Lambda$ given by $(s_U^{F,G})^\circ = s_{U^\circ}^{G,F}$.

Lemma 6.1. *Let F, G, H be finite groups. Let $I \in \mathcal{S}(F,G)$ and $J \in \mathcal{S}(G,H)$.*
(1) *Suppose I and J have the form $I = \Delta(A,1,\phi,1,B)$ and $J = \Delta(B,Y,\psi,Z,C)$. Then $t_I^{F,G} t_J^{G,H} = t_K^{F,H}$ where $K = \Delta(A,\phi(Y),\underline{\phi}\circ\psi,Z,C)$ and $\underline{\phi} : A/\phi(Y) \leftarrow B/Y$ is the isomorphism induced by ϕ.*
(2) *Suppose I and J have the form $I = \Delta(A,X,\phi,Y,B)$ and $J = \Delta(B,1,\psi,1,C)$. Then $t_I^{F,G} t_J^{G,H} = t_K^{F,H}$ where $K = \Delta(A,X,\phi\circ\underline{\psi},\psi^{-1}(Y),C)$ and $\underline{\psi}$ is induced by ψ.*

Proof. Let I, J, K be as in part (1). Then $K = I * J$. We have $I = \Delta(\phi)$ and $I^\circ = \Delta(\phi^{-1})$. So $I * I^\circ = \Delta(A)$ and $I^\circ * I = \Delta(B)$. Hence, $I^\circ * K = J$. Let $L \in \mathcal{S}(F,H)$ be such that $\tau_L^{I,J} \neq 0$. By part (2) of Theorem 5.2, $L \in \mathrm{ad}(K)$. The elements of $\mathcal{P}_L^{I,J}$ are the pairs (I,V) where $V \leq J$ and $L \leq I * V$. The condition on V can be expressed as $I^\circ * L \leq V \leq J$. By the defining formula, $\tau_L^{I,J} = \sum_V \text{möb}(V,J)$. Since $\tau_L^{I,J} \neq 0$, the recursive characterization of the Möbius function yields $I^\circ * L = J$, that is, $L = K$. Moreover, $\tau_K^{I,J} = \text{möb}(J,J) = 1$. Part (1) is established. Part (2) holds by a similar argument. Alternatively part (2) follows from part (1) by considering opposite morphisms. □

Lemma 6.2. *Given $B \leq G \in \mathcal{K}$, then $s_{\Delta(B)}^{G,G} = \sum_{Y \in \mathcal{S}(B)} t_{\Delta(Y)}^{G,G}$ as a sum of mutually orthogonal idempotents of* $\mathrm{End}_\Lambda(G)$.

Proof. This is immediate from the previous lemma. □

Lemma 6.3. *Given $A \leq F \in \mathcal{K} \ni G \geq B$, then $\{s_U^{F,G} : U \in \mathcal{S}(A,B)\}$ and $\{t_I^{F,G} : I \in \mathcal{S}(A,B)\}$ are $\mathbb{K}$-bases for $s_{\Delta(A)}^{F,F}.\Lambda(F,G).s_{\Delta(B)}^{G,G}$.*

Proof. By the previous two lemmas and part (3) of Theorem 5.2, $s_{\Delta(A)}^{F,F} t_{I'}^{F,G} s_{\Delta(B)}^{G,G} = 0$ for all $I' \in \mathcal{S}(F,G) - \mathcal{S}(A,B)$. The rest of the argument is easy. □

The next result supplies a technique for investigating the condition that Λ is semisimple.

Proposition 6.4. *Let $\mathcal{L}, \mathcal{M} \subseteq \mathcal{K}$, let $k : \mathcal{L} \leftarrow \mathcal{M}$ be a function and, for each $G \in \mathcal{M}$, let $\kappa_G : k(G) \leftarrow G$ be a group monomorphism. Then there is a corner embedding $\Lambda_{\mathcal{L}} \leftarrow \Lambda_{\mathcal{M}}$ given by $s^{k(F),k(G)}_{(\kappa_F \times \kappa_G)(U)} \leftarrow\!\shortmid s^{F,G}_U$ for $F, G \in \mathcal{M}$ and $U \in \mathcal{S}(F,G)$.*

Proof. By the previous lemma, the specified algebra monomorphism is a corner embedding. □

Corollary 6.5. *Given $G \in \mathcal{K}$, then the algebra $\mathrm{End}_{\mathbb{K}\mathcal{S}}(G) = \mathbb{K}\mathcal{S}(G,G)$ is semisimple if and only if G is trivial. In particular, the category algebra $\mathbb{K}\mathcal{S}_{\mathcal{K}}$ is locally semisimple if and only if every group in $\mathcal{K}$ is trivial.*

Proof. Putting $\ell = \mathrm{id}_{\mathbb{N}-\{0\}}$, then $\Lambda = k\mathcal{S}_{\mathcal{K}}$. If G is trivial then $\mathrm{End}_{\mathbb{K}\mathcal{S}}(G) \cong \mathbb{K}$. Suppose G is non-trivial and let A be a prime order subgroup of G. We may assume that $\mathcal{K} = \{A, G\}$. Putting $\mathcal{L} = \{G\}$ and $\mathcal{M} = \{A\}$, the latest proposition implies that $\mathrm{End}_{\mathbb{K}\mathcal{S}}(A)$ is isomorphic to a corner subalgebra of $\mathrm{End}_{\mathbb{K}\mathcal{S}}(G)$. In Example 3.7, we saw that $\mathrm{End}_{\mathbb{K}\mathcal{S}}(A)$ is not semisimple. □

The next two corollaries have similar proofs, which become easy after first noting that we can reduce to the case where $\mathcal{K}$ is finite.

Corollary 6.6. *Suppose $\mathcal{K}$ has an element H such that every element of $\mathcal{K}$ is isomorphic to a subgroup of H. Then Λ is locally semisimple if and only if $\mathrm{End}_\Lambda(H)$ is semisimple.*

Corollary 6.7. *Suppose, for all $F, G \in \mathcal{K}$, there exists $H \in \mathcal{K}$ such that F and G are isomorphic to subgroups of H. Then Λ is locally semisimple if and only if every object of Λ has a semisimple endomorphism algebra.*

We do not know whether, in analogy with [Rog19, 7.3], the hypothesis on $\mathcal{K}$ in the latest corollary can be dropped.

7 Semisimplicity and algebraic independence

We prove Theorem 1.1.

We define the **dual** of a Λ-module M to be the Λ-module M^* such that, given $F, G \in \mathcal{K}$, then $M^*(G)$ is the dual $\mathbb{K}$-module of $M(G)$ and, given $s \in \Lambda(F,G)$, then the action $s^\circ : M^*(G) \leftarrow M^*(F)$ is the adjoint of the action $s : M(F) \leftarrow M(G)$. When $\mathcal{K}$ is finite, the finite-dimensional $\mathbb{K}$-modules M^* and M can be regarded as mutual duals and, for any $s \in \Lambda$, the action of s° on M^* is the adjoint of the action of s on M. We also use a superscript $*$ to indicate the usual dual of a module of a group algebra over $\mathbb{K}$.

Theorem 3.6 gives the next lemma, and the subsequent lemma follows immediately.

Lemma 7.1. *Given an $\mathcal{S}$-seed (E, W) for $\mathcal{K}$, then $S^*_{E,W} \cong S_{E,W^*}$.*

Lemma 7.2. *Given a finite group E, then the simple Λ-modules having minimal group E are all projective if and only if they are all injective.*

For finite groups E and L, we write $\mathrm{epi}(E, L)$ to denote the set of group epimorphisms $E \leftarrow L$. For $\phi \in \mathrm{epi}(E, L)$, we write $\underline{\phi} : E \leftarrow L/\ker(\phi)$ for the isomorphism induced by ϕ. Identifying the codomain E of $\underline{\phi}$ with $E/1$, we define

$$\lhd(\phi) = \Delta(E, 1, \underline{\phi}, \ker(\phi), L) \in \mathcal{S}(E, L) , \qquad \rhd(\phi) = \Delta(L, \ker(\phi), \underline{\phi}^{-1}, 1, E) \in \mathcal{S}(L, E) .$$

Now suppose $\mathrm{epi}(E,L) \neq \emptyset$ and L is isomorphic to a subgroup of an element of $\mathcal{K}$. We define a square matrix T_E^L over $\mathbb{K}$, with rows and columns indexed by $\mathrm{epi}(E,L)$, such that, given $\phi, \psi \in \mathrm{epi}(E,L)$ then the (ϕ,ψ)-entry of T_E^L is

$$T_E^L(\phi,\psi) = \tau_{\Delta(E)}^{\triangleleft(\phi),\triangleright(\psi)}$$

In other words, to calculate T_E^L, we evaluate $t_{\triangleleft(\phi)}^{F,G} t_{\triangleright(\psi)}^{G,H}$ as a $\mathbb{K}$-linear combination of round basis elements, whereupon $T_E^L(\phi,\psi)$ is the coefficient of $t_{\Delta(E)}^{E,E}$. Note that the matrix T_E^L is well-defined up to conjugation by permutation matrices and, as such, T_E^L depends only on the isomorphism classes of E and L.

Theorem 7.3. *Suppose the matrix T_E^L is invertible for all finite groups L isomorphic to a subgroup of an element of $\mathcal{K}$ and all factor groups E isomorphic to a quotient group of L. Then Λ is locally semisimple.*

Proof. By Remark 2.3, we may assume that $\mathcal{K}$ is closed under subquotients up to isomorphism. (For those who prefer to work with finite-dimensional algebras, the same remark also allows us to assume that $\mathcal{K}$ is finite, though that step is not needed.) Let (E,W) be an $\mathcal{S}$-seed for $\mathcal{K}$. It suffices to show that the simple Λ-module $S_{E,W}$ is projective. By the closure property of $\mathcal{K}$, we can replace E with any isomorphic copy, so we may assume that $E \in \mathcal{K}$. Inductively, we may also assume that, given any $\mathcal{S}$-seed (E',W') such that E' is a strict factor group of E, then $S_{E',W'}$ is projective. Hence, by Lemma 7.2, each $S_{E',W'}$ is injective. Write $\mathcal{E} = \mathrm{End}_\Lambda(E)$. As usual, we regard W as a simple $\mathcal{E}$-module annihilated by the ideal $\mathcal{E}_<$. Let i be a primitive idempotent of $\mathcal{E}$ such that $iW \neq 0$. Then i is still primitive as an idempotent of Λ and $iS_{E,W} \neq 0$. So the indecomposable projective Λ-module $P = \Lambda i$ is the projective cover of $S_{E,W}$. We are to show that $P \cong S_{E,W}$. For a contradiction, suppose the unique maximal Λ-submodule Q of P is non-zero.

The $\mathcal{E}$-module $P(E) = \mathcal{E}i$ is the projective cover of W. Since $\mathbb{K}\mathrm{Aut}(E)$ is semisimple, $\mathcal{E}i/\mathcal{E}_< \cong W$, that is, $Q(E) \cong \mathcal{E}_<.i$. But the inductive assumption implies that, for any (E',W') as above, $S_{E',W'}$ cannot be a factor module of Q. Therefore, $Q(E) = 0$ and $\mathcal{E}_<$ annihilates $P(E)$.

Let G be of minimal order such that $Q(G) \neq 0$. Since $Q(G) \leq P(G) = \Lambda(G,E)i$, there exists $v \in \Lambda(G,E)$ such that $vi \in Q(G) - \{0\}$. Now v is a $\mathbb{K}$-linear combination of elements having the form $s_V^{G,E}$ with $V \in \mathcal{S}(G,E)$. Fixing V, then

$$s_V^{G,E} = s_V^{G,E} s_D^{E,E} / \ell(V_\bullet)$$

where $D = \Delta(V^\bullet, V_\bullet, \mathrm{id}, V^\bullet, V_\bullet)$. If the thorax $\Theta(V) \cong V^\bullet/V_\bullet \cong \Theta(D)$ is smaller than E, then $s_D^{E,E} \in \mathcal{E}_<$, hence $s_D^{E,E} i = 0$ and $s_V^{D,D} i = 0$. On the other hand, if $\Theta(V) \cong E$, then $V_\bullet = 1$ and $s_V^{G,E}$ is a $\mathbb{K}$-linear combination of elements $t_J^{G,E}$ where $J_\bullet = 1$. For such J, if $J^\bullet \neq E$, then $t_J^{G,E}$ is a $\mathbb{K}$-linear combination of elements $s_{V'}^{G,E}$ with $\Theta(V')$ smaller than E, whence each $s_{V'}^{G,E} i = 0$ and $t_J^{G,E} i = 0$. So we may assume that v is a $\mathbb{K}$-linear combination of elements having the form $t_J^{G,E}$ where $J^\bullet/J_\bullet = E/1$. Given $B \leq G$, then $t_{\Delta(B)}^{G,G} = t_{\Delta(B)}^{G,B} t_{\Delta(B)}^{B,G}$. By the minimality of G and the closure assumption on $\mathcal{K}$, if $B < G$ then $Q(B) = 0$, whence $t_{\Delta(B)}^{B,G} vi = 0$ and $t_{\Delta(B)}^{G,G} vi = 0$. So, by Lemma 6.3, $vi = t_{\Delta(G)}^{G,G} vi$. Hence, by part (3) of Theorem 5.2, we may assume that v is a $\mathbb{K}$-linear combination of elements $t_J^{G,E}$ where $J^\bullet/J_\bullet = E/1$ and ${}^\bullet J = G$. In other words,

$$v \in \bigoplus_{\psi \in \mathrm{epi}(E,G)} \mathbb{K}\, t_{\triangleright(\psi)}^{G,E} .$$

Since $\{t_I^{E,E} : I \in \mathcal{S}(E,E)_<\}$ is a $\mathbb{K}$-basis for $\mathcal{E}_<$, any $w \in \mathcal{E}$ can be expressed uniquely as $w = w_< + w_=$ where $w_< \in \mathcal{E}_<$ and

$$w_= = \sum_{\epsilon \in \mathrm{Aut}(E)} \partial_{\Delta(\epsilon)}(w) t^{E,E}_{\Delta(\epsilon)}$$

with each $\partial_{\Delta(\epsilon)}(w) \in \mathbb{K}$. Every simple factor of the Λ-module $\Lambda\mathcal{E}_<$ is isomorphic to $S_{E',W'}$ for some (E',W') as above. Since none of the simple factors of Q have that form, $Q \cap \Lambda\mathcal{E}_< = 0$. So $vi_< \notin Q - \{0\}$. But $vi \in Q - \{0\}$. Therefore, $vi_= \neq 0$. We have

$$t^{G,E}_{\triangleright(\psi)} t^{E,E}_{\Delta(\epsilon)} = t^{G,E}_{\triangleright(\epsilon^{-1}\psi)}$$

by Lemma 6.1. So we can write

$$vi_= = \sum_{\psi \in \mathrm{epi}(E,G)} v(\psi) t^{G,E}_{\triangleright(\psi)}$$

with each $v(\psi) \in \mathbb{K}$. Note that $v(\psi) \neq 0$ for some ψ. Let $(u(\phi) : \phi \in \mathrm{epi}(E,G))$ be any family of elements $u(\phi) \in \mathbb{K}$. Write

$$u = \textstyle\sum_\phi u(\phi) t^{E,G}_{\triangleleft(\phi)} \ .$$

Since $vi \in Q(G)$ and $\Lambda(E,G)Q(G) = Q(E) = 0$, we have $uvi = 0$. So $uvi_= = -uvi_< \in \mathcal{E}_<$. Therefore,

$$0 = \partial_{\Delta(E)}(uvi_=) = \sum_{\phi,\psi \in \mathrm{epi}(E,G)} u(\phi) T^G_E(\phi,\psi) v(\psi) \ .$$

But the vector $(u(\phi) : \phi)$ is arbitrary, the vector $(v(\psi) : \psi)$ is non-zero and the matrix T^G_E is invertible. We have obtained a contradiction, as required. □

To apply the theorem, we shall be needing another lemma.

Lemma 7.4. *Let A, B, C be subgroups of finite groups F, G, H, respectively. Let $\phi : A \leftarrow B$ and $\psi : C \leftarrow B$ be group epimorphisms and $\theta : A \leftarrow C$ a group isomorphism. Write $I = \triangleleft(\phi)$, $J = \triangleright(\psi)$, $K = \Delta(\theta)$. Suppose $K \leq I * J$. Then (I,J) is compatible and $K \in \mathrm{ad}(I * J)$. We have $I * J = (\theta \times \psi)(B)$. Letting S run over those subgroups of B such that $K \leq (\phi \times \psi)(S)$, then*

$$\tau^{I,J}_K = \sum_S \mathrm{m\ddot{o}b}(S,B) \ell(\ker(\phi) \cap S \cap \ker(\psi)) \ .$$

Furthermore, if $\ker(\phi) = \ker(\psi)$, then $K = I * J$ and $t^{F,G}_I t^{G,H}_J = \tau^{I,J}_K t^{F,H}_K$.

Proof. The preambulatory parts and the rider are clear. We need only prove the formula for $\tau^{F,G}_K$. We shall apply part (4) of Theorem 5.2. Write $M = \ker(\phi)$ and $N = \ker(\psi)$. Let (R,T) run over the pairs of subgroups $R \leq B \geq T$ such that $K \leq (\phi \times \psi)(R \cap T)$. The discussion of the star product in Section 2 shows that $\mathcal{P}^{I,J}_K$ is the set of pairs having the form (U_R, V_T), where

$$U_R = \triangleleft(\phi_R) = \Delta(A, 1, \underline{\phi_R}, M \cap R, R) \ , \qquad V_T = \triangleright(\psi_T) = \Delta(T, T \cap N, \underline{\psi_T}^{-1}, 1, C)$$

with $\underline{\phi_R}$ denoting the isomorphism induced by the restriction $\phi_R : A \leftarrow R$ of ϕ, similarly for $\underline{\psi_T}$ and $\psi_T : C \leftarrow T$. The retraction $\rho^{I,J}_K$ in the proof of Theorem 5.2 is given by

$$\rho^{I,J}_K(U_R, V_T) = ((\Delta(A, 1, \phi_S, M \cap S, S), \Delta(S, S \cap N, \underline{\psi_S}^{-1}, 1, C))$$

where $S = R \cap T$. The pairs having the form on the left-hand side are precisely the elements of the image $\mathcal{R}^{I,J}_K$ of $\rho^{I,J}_K$, moreover, the subgroups S arising in this way are precisely the subgroups S specified in the assertion. □

Suppose ℓ is algebraically independent with respect to $\mathcal{K}$. Let $\Pi_{\mathcal{K}}$ be the set of prime divisors of the orders of the elements of $\mathcal{K}$. For each $q \in \Pi_{\mathcal{K}}$, write $\lambda_q = \ell(q)$. The assumption on ℓ is that the elements $\lambda_q \in \mathbb{K}$ are algebraically independent. Let $\mathcal{O}$ be the integral domain generated over $\mathbb{Q}$ by the λ_q. Any $\mathfrak{o} \in \mathcal{O}$ can be expressed uniquely as a polynomial expression in $(\lambda_q : q \in \Pi_{\mathcal{K}})$ with coefficents in $\mathbb{Q}$, so we may speak of the **degree** of $\mathfrak{o}$. We call $\mathfrak{o}$ **monic** provided the associated polynomial expression has a unique term of maximal degree and the coefficient for that term is 1. For a positive integer n, we write $\mathrm{len}(n)$ for the length of n, we mean, the number of prime factors up to multiplicity.

We now prove Theorem 1.1. It suffices to show that, in the notation of Theorem 7.3, the matrix T_E^L is invertible. We shall show that, in fact, the determinant of T_E^L is a monic element of $\mathcal{O}$ with degree $d^{|\mathrm{epi}(E,L)|}$, where $d = \mathrm{len}(|L|/|E|)$. As before, we may assume that $E, L \in \mathcal{K}$. Let $\phi \in \mathrm{epi}(E, L)$. By Lemma 7.4,

$$T_E^L(\phi,\phi) = \tau_{\Delta(E)}^{\lhd(\phi),\rhd(\phi)} = \sum_{S \in \mathcal{S}(B) : \Delta(E) \leq \phi(S)} \mathrm{möb}(S,B)\ell(\ker(\phi) \cap S)$$

which is a monic element of $\mathcal{O}$ with degree $\mathrm{len}(|\ker(\phi)|) = d$. Let $\psi \in \mathrm{epi}(E, L) - \{\phi\}$. It remains only to show that the off-diagonal matrix entry

$$T_E^L(\phi,\psi) = \tau_{\Delta(E)}^{\lhd(\phi),\rhd(\psi)}$$

is an element of $\mathcal{O}$ with degree less than d. We may assume that $T_E^L(\phi,\psi) \neq 0$. Then, by part (2) of Theorem 5.2, $\Delta(E) \leq \lhd(\phi) * \rhd(\psi)$. Bearing in mind that $\phi \neq \psi$, the rider of Lemma 7.4 implies that $\ker(\phi) \neq \ker(\psi)$. By the formula in the lemma, $T_E^L(\phi,\psi)$ has degree at most $\mathrm{len}(|\ker(\phi) \cap \ker(\psi)|) < d$. The proof of Theorem 1.1 is complete.

8 The trivial module

We define the **trivial Λ-module** to be the isomorphically unique simple Λ-module $S_{1,1}$ whose minimal groups are trivial. In this section, we give some criteria for $S_{1,1}$ to be projective and injective.

The set $\mathcal{K}$ is finite if and only if the algebra Λ is unital. When that condition holds, Λ is semisimple if and only if all the block algebras of Λ are simple. For finite $\mathcal{K}$, we define the **principal block** of Λ to be the block $b_{1,1}$ of Λ that fixes $S_{1,1}$. We call $\Lambda b_{1,1}$ the **principal block algebra**.

Lemma 8.1. *The trivial Λ-module $S_{1,1}$ is projective if and only if $S_{1,1}$ is injective. When $\mathcal{K}$ is finite, that condition holds if and only if the principal block algebra $\Lambda b_{1,1}$ is simple.*

Proof. This follows from Lemma 7.2. □

In the context of the biset category, Rognerud [Rog19, 7.6] noted that, for finite $\mathcal{K}$, an analogous simple module of $\mathbb{K}B_{\mathcal{K}}$ (again called the trivial module) is projective if and only if $\mathbb{K}B_{\mathcal{K}}$ is semisimple. That raises the question as to whether a similar assertion holds for Λ.

Regarding ℓ as a formal function on finite groups, let φ denote the totient functon for ℓ. That is to say, φ is the isomorphism invariant formal function on finite groups determined by the equivalent conditions

$$\ell(U) = \sum_{I \in \mathcal{S}(U)} \varphi(I) , \qquad \varphi(I) = \sum_{U \in \mathcal{S}(I)} \mathrm{möb}(U,I)\ell(U)$$

where U is a given finite group in the first equation, I likewise in the second. The next result, an observation made by Hall [Hal36], can be seen straight away by considering the subgroup generated by a given d-tuple of elements of a given finite group.

Proposition 8.2. (Philip Hall.) *Let d be a positive integer. Suppose $\ell(n) = n^d$ for all positive integers n. Given a finite group G, then $\varphi(G)$ is the number of d-tuples $(g_1, ..., g_d)$ such that $\{g_1, ..., g_d\}$ is a generating set for G. In particular, $\varphi(G)$ is non-zero if and only if the minimal number of generators of G is at most d.*

When $d = 1$, we have $\varphi(G) = 0$ unless G is cyclic, in which case, $\varphi(G) = \phi(|G|)$, where ϕ is the Euler totient function.

We present the next result as a separate application of Lemma 5.1, but it can also be seen as a specialization of material implicit in the proof of Boltje-Danz [BD13, 4.2].

Lemma 8.3. *Given a finite group G, then*

$$\varphi(G) = \sum_{M,N \in \mathcal{S}(G)} \text{möb}(M,G)\text{möb}(N,G)\ell(M \cap N) \, .$$

Proof. Consider the poset $\mathcal{P} = \mathcal{S}(G) \times \mathcal{S}(G)$. Let $\ell_{\mathcal{P}} : \mathbb{Z} \leftarrow \mathcal{P}$ be the function given by $\ell_{\mathcal{P}}(M,N) = \ell(M \cap N)$ for $(M,N) \in \mathcal{P}$. The retraction $(M \cap N, M \cap N) \leftarrow\!\shortmid (M,N)$ of $\mathcal{P}$ preserves $\ell_{\mathcal{P}}$ and has image $\mathcal{P}' = \{(L,L) : L \in \mathcal{S}(G)\}$. Since $\text{möb}_{\mathcal{P}'}((K,K),(L,L)) = \text{möb}(K,L)$ for $K, L \leq G$, the required conclusion follows from Lemma 5.1. □

Lemma 8.4. *Let F, G, H be finite groups and $B \leq G$. Then $t^{F,G}_{1\times B} t^{G,H}_{B\times 1} = \varphi(B) t^{F,H}_{1\times 1}$.*

Proof. This follows from Lemmas 7.4 and 8.3. □

For arbitrary $\mathcal{K}$, let $P_{1,1}$ denote the projective cover of $S_{1,1}$. Given $G \in \mathcal{K}$, let $i_G = s^{G\times G}_{1\times 1} = t^{G\times G}_{1\times 1}$. Using Theorem 5.2, it is easy to show that i_G is primitive as an idempotent of $\text{End}_\Lambda(G)$ and hence also as an idempotent of Λ. So Λi_G is an indecomposable projective Λ-module.

Lemma 8.5. *With the notation above, $\Lambda i_G \cong P_{1,1}$ as Λ-modules.*

Proof. By Proposition 2.1, we may assume that $\mathcal{K}$ owns a trivial group 1. Also writing 1 to denote the trivial subgroup of G then, by considering the elements $s^{1\times G}_{1\times 1}$ and $s^{G\times 1}_{1\times 1}$, we see that the idempotents i_G and i_1 are associate. Therefore, $i_G S_{1,1} \neq 0$. □

Theorem 8.6. *Suppose $\varphi(B) \neq 0$ for every subgroup B of every element of $\mathcal{K}$. Then $S_{1,1}$ is projective and injective, furthermore, if $\mathcal{K}$ is finite, then the algebra $\Lambda b_{1,1}$ is simple.*

Proof. By Lemma 8.1, it suffices to show that $P_{1,1}$ is simple. Let Q be a non-zero submodule of $P_{1,1}$. Let $G \in \mathcal{K}$ such that $Q(G) \neq 0$. Lemma 8.5 allows us to put $P_{1,1} = \Lambda i_G$. If we can show that $i_G \in Q(G)$, then it will follow that $Q = P_{1,1}$, and the simplicity of $P_{1,1}$ will be established. Let $y \in Q(G) - \{0\}$. By part (3) of Theorem 5.2,

$$y = \sum_{B \in \mathcal{S}(G)} y_B t^{G,G}_{B,1}$$

with each $y_B \in \mathbb{K}$. Choose B such that $y_B \neq 0$. By Lemma 8.4, $t^{G,G}_{1,B} y = y_B \varphi(B) i_G$ Therefore, $i_G \in Q(G)$, as required. □

The theorem together with Proposition 8.2 yields a corollary.

Corollary 8.7. *Let d be a positive integer. Suppose $\ell(n) = n^d$ for all positive integers n. Suppose also that every subgroup of every element of $\mathcal{K}$ has minimal number of generators at most d. Then the conclusion of Theorem 8.6 holds.*

9 Deformation of the biset category

We shall introduce a small $\mathbb{K}$-linear category Γ on $\mathcal{K}$, in other words, an algebra Γ over $\mathbb{K}$ equipped with a complete family of mutually orthogonal idempotents $(\mathrm{id}_G^\Gamma : G \in \mathcal{K})$. Again, Γ will be determined by $\mathcal{K}$, $\mathbb{K}$, ℓ. A case of motivating importance is that where ℓ is the inclusion of the positive integers in $\mathbb{K}$, we mean to say, $\ell(n) = n$ for all positive integers n. In that case, we shall find that Γ coincides with $\mathbb{K}\mathcal{B}_{\mathcal{K}}$, the $\mathbb{K}$-linear biset category on $\mathcal{K}$. That will justify interpretation of Γ as a deformation of $\mathbb{K}\mathcal{B}_{\mathcal{K}}$.

Let us briefly summarize the construction of $\mathbb{K}\mathcal{B}_{\mathcal{K}}$ and some other related categories, starting with the **biset category** $\mathcal{B}$. For a full discussion, see Bouc [Bou10, Chapters 2, 3]. The objects of $\mathcal{B}$ are the finite groups. Consider finite groups F, G, H. An F**-**G**-biset** is defined to be an $F{\times}G$-set with the action of F written on the left, the action of G written on the right. For an F-G-biset X and a G-H-biset Y, we write $X \times_G Y$ to denote the set of G-orbits of $X{\times}Y$, and we regard $X \times_G Y$ as an F-H-biset. The morphism set $\mathcal{B}(F,G)$ the Grothendieck group of the category of F-G-bisets, with relations such that addition is induced by disjoint union. Thus, $\mathcal{B}(F,G)$ has a $\mathbb{Z}$-basis consisting of the isomorphism classes $[(F{\times}G)/U]$ of the F-G-bisets having the form $(F{\times}G)/U$, where $U \in \mathcal{S}(F,G)$. Note that $[(F{\times}G)/U]$ is uniquely determined by the conjugacy class of U. The composition operation $\mathcal{B}(F,H) \leftarrow \mathcal{B}(F,G) \times \mathcal{B}(G,H)$ is the $\mathbb{Z}$-linear map induced by the operation $\times_G$ on bisets. Explicitly, [Bou10, 2.3.34] tells us that, for $U \in \mathcal{S}(F,G)$ and $V \in \mathcal{S}(G,H)$, we have

$$\left[\frac{F{\times}G}{U}\right]\left[\frac{G{\times}H}{V}\right] = \sum_{U^{\bullet}.g.{}^{\bullet}V \subseteq G} \left[\frac{F{\times}H}{U * {}^{g\times 1}V}\right]$$

where the notation indicates that g runs over representatives of the double cosets of $U^{\bullet}$ and ${}^{\bullet}V$ in G.

We define the $\mathbb{K}$**-linear biset category** $\mathbb{K}\mathcal{B}$ to be the $\mathbb{K}$-linear category whose objects are the finite groups, the morphism $\mathbb{K}$-module $\mathbb{K}\mathcal{B}(F,G)$ is the $\mathbb{K}$-linear extension of $\mathcal{B}(F,G)$ and the composition for $\mathbb{K}\mathcal{B}$ is the $\mathbb{K}$-linear extension of the composition for $\mathcal{B}$. Note that $\mathbb{K}\mathcal{B}$ is not to be confused with the $\mathbb{K}$-linearization of $\mathcal{B}$, which could be expressed with the same notation. As a direct sum of regular $\mathbb{K}$-modules,

$$\mathbb{K}\mathcal{B}(F,G) = \bigoplus_{U \in_{F\times G} \mathcal{S}(F,G)} \mathbb{K}\left[\frac{F{\times}G}{U}\right]$$

with the notation indicating that U runs over representatives of the conjugacy classes of subgroups of $F{\times}G$.

We introduce a $\mathbb{K}$-linear category $\mathbb{K}_\sigma\mathcal{B}$, defined as follows. The objects of $\mathbb{K}_\sigma\mathcal{B}$ are the finite groups. The morphism $\mathbb{K}$-module $\mathbb{K}_\sigma\mathcal{B}(F,G)$ has a formal $\mathbb{K}$-basis $\{d_U^{F,G} : U \in_{F\times G} \mathcal{S}(F,G)\}$. We define the composition to be such that

$$d_U^{F,G} d_V^{G,U} = \sum_{U^{\bullet}.g.{}^{\bullet}V \subseteq G} \frac{\ell(U_{\bullet} \cap {}^{g}({}_{\bullet}V))}{|U_{\bullet} \cap {}^{g}({}_{\bullet}V)|} \, d^{F,H}_{U * {}^{g\times 1}V} \, .$$

Some properties need to be checked. It is not hard to see that the right-hand expression is well-defined in that the value does not change when U and V are replaced by conjugate subgroups or when a different choice is made for the double coset representatives g. We postpone, to Theorem 9.3, proof of the associativity of the composition for $\mathbb{K}_\sigma\mathcal{B}$. It is clear already that, if associativity does hold, then $\mathbb{K}_\sigma\mathcal{B}$ is a $\mathbb{K}$-linear category whose identity morphism on G is

$d^{G,G}_{\Delta(G)}$. It is also clear already that, in the motivating case mentioned above, where ℓ is the inclusion of the positive integers, $\mathbb{K}_\sigma\mathcal{B}$ is indeed a $\mathbb{K}$-linear category and, in fact, $\mathbb{K}_\sigma\mathcal{B}$ can be identified with $\mathbb{K}\mathcal{B}$ by identifying each $d^{F,G}_U$ with $[(F{\times}G)/U]$.

Now let us discuss the category $\mathbb{K}_\sigma\mathcal{S}$. We make $\mathbb{K}_\sigma\mathcal{S}(F,G)$ become an $\mathbb{F}(F{\times}G)$-module via the conjuation action of $F{\times}G$ on $\mathcal{S}(F,G)$. That is to say,

$$ {}^{f\times g}s^{F,G}_U = s^{F,G}_{{}^{f\times g}U} \ . $$

To describe the action in another way, we introduce the unity-preserving algebra map $\sigma_G : \mathbb{K}_\sigma\mathcal{S}(G,G) \leftarrow \mathbb{K}G$ given by $\sigma_G(g) = s^{G,G}_{\Delta(G,g,G)}$ where $\Delta(G,g,G) = \{{}^g b \times b : b \in G\}$.

Remark 9.1. *With the notation above, given $x \in \mathbb{K}_\sigma\mathcal{S}(F,G)$, then ${}^{f\times g}x = \sigma_F(f).x.\sigma_G(g^{-1})$.*

Proof. By $\mathbb{K}$-linearity, we may assume that $x = s^{F,G}_U$, whereupon the verification is an easy calculation. □

Since $\mathbb{K}G$ is semisimple, the principal block of $\mathbb{K}G$ is $e_G = \sum_{g\in G} g/|G|$. We define

$$ \overline{s}^{F,G}_U = \sigma_F(e_F).s^{F,G}_U.\sigma_G(e_G) = \frac{1}{|F|.|G|} \sum_{f\in F, g\in G} s^{F,G}_{{}^{f\times g}U} \ . $$

Plainly, $\overline{s}^{F,G}_U$ depends only on the $F{\times}G$-conjugacy class of U. We define a category $\overline{\mathbb{K}_\sigma\mathcal{S}}$ such that the objects are the finite groups and the $\mathbb{K}$-module of morphisms $F \leftarrow G$ is the $F{\times}G$-fixed submodule

$$ \overline{\mathbb{K}_\sigma\mathcal{S}}(F,G) = (\mathbb{K}_\sigma\mathcal{S}(F,G))^{F\times G} = \bigoplus_{U \in_{F\times G} \mathcal{S}(F,G)} \mathbb{K}\overline{s}^{F,G}_U \ . $$

It is easy to check that $\overline{\mathbb{K}_\sigma\mathcal{S}}$ is a $\mathbb{K}$-linear category whose identity morphism on G is the element $\sigma_G(e_G) = \overline{s}^{G,G}_{\Delta(G)}$. The next theorem describes how $\mathbb{K}_\sigma\mathcal{B}$ can be identified with $\overline{\mathbb{K}_\sigma\mathcal{S}}$. First, we need a lemma.

Lemma 9.2. *Let F, G, H be finite groups. Let $U \in \mathcal{S}(F,G)$ and $V \in \mathcal{S}(G,H)$. Then*

$$ |U|.|V| = |U^\bullet.{}^\bullet V|.|U_\bullet \cap {}_\bullet V|.|U * V| \ . $$

Proof. Let $A = \{f{\times}g{\times}h : f{\times}g \in U, g{\times}h \in V\}$. Note that $U^\bullet \cap {}^\bullet V = \{g : f{\times}g{\times}h \in A\}$. Fix $f{\times}g{\times}h \in A$. Given $f' \in F$ and $h' \in H$, then $f'{\times}g{\times}h' \in A$ if and only if $f'f^{-1} \in {}_\bullet U$ and $h'h^{-1} \in V$. So

$$ |A| = |U^\bullet \cap {}^\bullet V|.|{}_\bullet U|.|V_\bullet| \ . $$

Given $g' \in G$, then $f{\times}g'{\times}h \in A$ if and only if $g'g^{-1} \in U_\bullet \cap {}_\bullet V$. So

$$ |A| = |U_\bullet \cap {}_\bullet V|.|U * V| \ . $$

Eliminating $|A|$ and using $|U| = |U^\bullet|.|{}_\bullet U|$ and $|V| = |{}^\bullet V|.|V_\bullet|$, we obtain

$$ |U|.|V|.|U^\bullet \cap {}^\bullet V| = |U^\bullet|.|{}^\bullet V|.|U_\bullet \cap {}_\bullet V|.|U * V| \ . $$

□

We let $\nu^{F,G} : \overline{\mathbb{K}_\sigma\mathcal{S}}(F,G) \leftarrow \mathbb{K}_\sigma\mathcal{B}(F,G)$ be the $\mathbb{K}$-linear isomorphism given by

$$ \nu^{F,G}(d^{F,G}_U) = |G|\,\overline{s}^{F,G}_U/|U| \ . $$

Let us mention that another way of expressing the formula is $\nu^{F,G}(d^{F,G}_U) = \mathrm{tr}^{F,G}_U(s^{F,G}_U)/|F|$, where $\mathrm{tr}^{F,G}_U$ denotes the transfer map from the U-fixed submodule.

Theorem 9.3. *The composition operation on $\mathbb{K}_\sigma\mathcal{B}$ is associative, $\mathbb{K}_\sigma\mathcal{B}$ is a $\mathbb{K}$-linear category and the maps $\nu^{F,G}$ determine an isomorphism of $\mathbb{K}$-linear categories $\nu : \overline{\mathbb{K}_\sigma\mathcal{S}} \leftarrow \mathbb{K}_\sigma\mathcal{B}$ that acts on objects as the identity.*

Proof. Letting F, G, H, U, V be as above, we must show that

$$\nu^{F,G}(d_U^{F,G})\,\nu^{G,H}(d_V^{G,H}) = \nu^{F,H}(d_U^{F,G}d_V^{G,H})\;.$$

Applying Remark 9.1 to evaluate $\sigma_G(e_G)s_V^{G,H}$, we obtain

$$s_U^{F,G}\sigma_G(e_G)s_V^{G,H} = \frac{1}{|G|}\sum_{g\in G}\sigma(U,{}^{g\times1}V)\,s^{F,H}_{U*{}^{g\times1}V}\;.$$

Since $\sigma(U,{}^{g\times1}V) = \ell(U_\bullet\cap{}^g({}_\bullet V))$, which depends only on $U^\bullet.g.{}^\bullet V$, we have

$$\overline{s}_U^{F,G}\,\overline{s}_V^{G,H} = \frac{1}{|G|}\sum_{U^\bullet.g.{}^\bullet V\subseteq G}|U^\bullet.g.{}^\bullet V|\,\ell(U_\bullet\cap{}^g({}_\bullet V))\,\overline{s}^{F,H}_{U*{}^{g\times1}V}\;.$$

Lemma 9.2 now yields

$$\frac{\overline{s}_U^{F,G}\,\overline{s}_V^{G,H}}{|U|.|V|} = \frac{1}{|G|}\sum_{U^\bullet.g.{}^\bullet V\subseteq G}\frac{\ell(U_\bullet\cap{}^g({}_\bullet V))}{|U_\bullet\cap{}^g({}_\bullet V)|}\cdot\frac{\overline{s}^{F,H}_{U*{}^{g\times1}V}}{|U*{}^{g\times1}V|}\;.$$

Multiplying by $|G|.|H|$ gives the required equality. □

The isomorphism ν does not preserve the antiisomorphisms induced by taking opposite subgroups. We mention that, if $\mathbb{K}$ owns square roots of the orders of all the groups in $\mathcal{K}$, then ν can be replaced by an isomorphism, described as follows, which does have that symmetry property. For each G, we arbitrarily choose a square root $\sqrt{|G|}$ of $|G|$. We then replace $\nu^{F,G}$ with the map $\sqrt{|F|.|G|}\,\overline{s}_U^{F,G}/|U| \hookleftarrow d_U^{F,G}$.

We now pass to a full subcategory. We define

$$\Gamma = \mathbb{K}_\sigma\mathcal{B}_\mathcal{K}$$

as a small $\mathbb{K}$-linear category and as an algebra over $\mathbb{K}$. By a comment earlier in this section, when ℓ is the inclusion of the positive integers, we can identify $\Gamma = \mathbb{K}\mathcal{B}_\mathcal{K}$ by identifying $d_U^{F,G} = [(F\times G)/U]$ for each $F\in\mathcal{K}\ni G$ and $U\in\mathcal{S}(F,G)$. The latest theorem tells us that Γ is isomorphic to a corner subalgebra of Λ. Corollary 1.3 follows.

The same theorem also shows that, if every group in $\mathcal{K}$ is abelian, then $\Gamma\cong\Lambda$. Now suppose that every group in $\mathcal{K}$ is cyclic. To complete the proof of Theorem 1.4, we must show that Λ is locally semisimple. Let E and L be cyclic groups such that $|E|$ divides $|L|$ which in turn divides the order of an element of $\mathcal{K}$. By Theorem 7.3, it suffices to show that the matrix T_E^L is invertible. Let $\phi,\psi\in\mathrm{epi}(E,L)$. Let M be the unique subgroup of L such that $|M| = |L|/|E|$. Then $\ker(\phi) = M = \ker(\psi)$. By Lemma 7.4

$$t^{E,L}_{\triangleleft(\phi)}t^{L,E}_{\triangleright(\psi)} = \tau^{E,E}_{\Delta(\theta)}t^{E,E}_{\Delta(\theta)}$$

where $\theta = \phi\circ\psi^{-1}$. If $\phi=\psi$, then $\Delta(\theta)=\Delta(E)$ and

$$T_E^L(\phi,\psi) = \tau^{E,E}_{\Delta(E)} = \ell(M)\neq 0\;.$$

If $\phi\neq\psi$, then $\Delta(\theta)\neq\Delta(E)$ and $T(\phi,\psi)=0$. So T_E^L is a non-zero multiple of the identity matrix. The proof of Theorem 1.4 is complete.

References

[Bar08] L. Barker, *Rhetorical biset functors, rational p-biset functors and their semisimplicity in characteristic zero*, J. Algebra **319**, 3810-3853 (2008).

[BD16] L. Barker, M. Demirel, *Simple functors of admissible linear categories*, Algebras and Representation Theory **19**, 463-472 (2016).

[BO] L. Barker, İ. A. Öğüt, *Some deformations of the fibred biset category*, (in preparation).

[BD13] R. Boltje, S. Danz, *A ghost algebra of the double Burnside algebra in characteristic zero*, J. Pure Applied Algebra **217**, 608-635 (2013).

[BX08] R. Boltje, B. Xu, *On p-permutation equivalences: between Rickard equivalences and isotypies*, Trans. AMS, **360**, 5067-5087 (2008).

[Bou10] S. Bouc, "Biset Functors for Finite Groups", Lecture Notes in Math. vol. 1990 (Springer, Berlin, 2010).

[BST13] S. Bouc, R. Stancu, J. Thévenaz, *Simple biset functors and double Burnside ring*, J. Pure Applied Algebra **217**, 546-566 (2013).

[BT] S. Bouc, J. Thévenaz, *Tensor product of correspondence functors*, J. Algebra (to appear).

[Duc16] M. Ducellier, *A study of a simple p-permutation functor*, J. Algebra **447**, 367-382 (2016).

[Gre07] J. A. Green, "Polynomial Representations of GL_n", 2nd ed., Lecture Notes in Math. 830, (Springer, Berlin, 2007).

[Hal36] P. Hall, *The Eulerian functions of a group*, Quart. J. Math. (Oxford) **7**, 134-151 (1936).

[Lin04] M. Linckelmann, *Fusion category algebras*, J. Algebra **227**, 222-235 (2004).

[Rog19] B. Rognerud, *Around evaluations of biset functors*, **69**, 805-843 (2019).

[Sta11] R. P. Stanley, "Enumerative Combinatorics, Vol. 1", 2nd ed., (Cambridge Univ. Press).

[Sy888] J. J. Sylvester, *Note on a proposed addition to the vocabulary of arithmetic*, Nature **37**, 152-153 (1888).

[TW95] J. Thévenaz, P. J. Webb, *The structure of Mackey functors*, Trans. Amer. Math. Soc. **347**, 1865-1961 (1995).