Eudoxus, the origin of reasoning by creation and subtraction?

Laurence Barker, 6 November 2013, Bilkent University.

This is a recreational essay, references omitted, details neglected. My reading of history and philosophy being hapazard, I am not sure how widely familiar the speculations might be.

Abstract: The logic of mathematics is of interest as an extreme case of the logic of science which, in turn, is an application of logic in general. In mathematics, we reason just as natural scientists do, except that, in our final presentations, all the reasoning is removed except for definition and deduction. This style of reasoning though definition and deduction, was fully established by the time of Aristotle: "The definition is the logos of the essence". We shall speculate that it originated with Eudoxus. We hope that, whether or not our speculation is sound, our illustration will help to indicate something of the nature of the art of definition.

Creation and subtraction, in other words, definition and deduction

Definitions are a vital aspect of reasoning. If one wishes to make a study of the art of definition, then its role in the logic of pure mathematics might be a good place to start.

I confess, I am not sure what I mean, or should mean, by *logic of pure mathematics*. (The art of definition is very difficult.) Let us lean towards *logic* in the narrowest possible sense, *logic of perfect exposition*: **ideal exposition when an investigation is complete**; the system of constraints governing the way, ideally, one person communicates finished material to another.

An ideal exposition in pure mathematics would have at least one theorem. Typically, theorems are succinct.

Example theorem: *Every Euclidian domain is a principal ideal domain.*

What would one have to cover to convey this theorem to an audience? Equivalently, what would one have to assimilate to know this theorem?

Certainly, an exposition of the theorem would have to include a proof. But it would also have to include statements of the definitions and justifications of the definitions. To justify them, there would have to be a discussion of motivations.

In almost every mathematical seminar, research paper, lecture, textbook (except at a very introductory level), much of the time and effort is spent on setting up definitions.

So, even if we understand *logic* in the narrowest possible sense, the logic of pure mathematics must include at least two components:

Creation: One way to create concepts, though not the only way, is through the tricky art of definition. (It is methodologically tricky. We do not normally teach it to undergraduates.)
Subtraction, deduction: The theorem is justified using the straightforward craft of proof. (A proof is a very clear deductive explanation. We do normally teach it to undergraduates.)

We shall be focussing on the tricky art component, not the straightforward craft component.

Let us try to seek out the origin of creative mathematics. In the second half of the 4th century BC, in *Metaphysics Book VII*, Aristotle wrote

> "The definition of a substance is the logos of the essence."

Stripping away the Platonic metaphysics, we can interpret it to mean: **by defining a thing, we express its fundamental nature and we make the thing subject to reason.**

The attitude, here, goes back at least as far as Plato. We shall be commenting on Plato's dialogue *Meno* at the end of this essay. We shall be concluding, speculatively, that Aristotle's and Plato's emphasis on definition was derived from some mathematics attributed to Eudoxus.

The logic of pure mathematics tends to reverse chronology

Let us issue a general warning about the pure mathematical literature. The literature does not give clear testimony to the whole methodology. Passage from imperfect exposition to perfect exposition involves drastic reorganization, ruthlessly eliminating all heuristics.

Let us also issue one rather particular warning. Theory, in pure mathematics, normally develops in the following three stages. Of course, the stages overlap and interact. We mean only to give a rough impression of the flow.

A: An idea arises in sophisticated scenarios where use of the idea is crucial.

B: The idea is abstracted and a theory consisting of definitions and proofs is established.

C: Further applications include pedagogical reassessment of well-understood easy scenarios.

When the material has settled, it tends to be reorganized as follows.

BC: Introductory texts present the theory and some easy applications.

A: Specialist texts discuss the more sophisticated applications.

Later mathematicians, including students, may surmise thus:

C: presumably, the easy applications stimulated the ideas,

B: and then, presumably, the theory must have been developed,

A: and then, presumably, the sophisticated applications must have emerged.

As a consequence:

General Law of History of Mathematics: Surmised chronological order of progress, gleaned from the technical literature, tends to be the reverse of actual chronological order.

Example: At present, the core undergraduate algebra syllabus begins in the early 20th and late 19th centuries, then works backwards through the 19th century.

Euclid's treatment of Pythagoras' Theorem

If nothing else, the material in the present section does at least serve the pedagogical purpose of illustrating the principle of logic that Aristotle was enunciating in "The definition ... is the logos of the essence".

We shall be giving a shamelessly anachronistic sketch of some material in Euclid's *Elements*. We shall be making use of modern notions of numbers, decimal notation and algebraic notation. My excuse for the anachronism is that I wish to concisely convey, to a modern audience, **how some trouble was resolved using some brilliant definitions**; though the definitions seem likely to have been introduced for other purposes.

Elements closely conforms to Aristotle's views on logic. Most of its thirteen books begin with a pile of definitions. *Book 1* opens with "A point is that which has no part. A line is breadthless length. The ends of a line are points."

Euclid presents a long and complicated proof of Pythagoras' Theorem in *Book 1.* After giving an account of the theory of ratios in *Book 5*, he makes use of that theory to supply another proof of the theorem in *Book 6.*

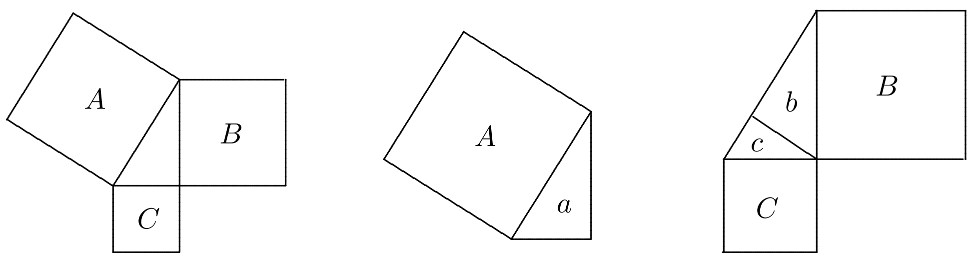
The concepts involved in the statement of Pythagoras' theorem are not easy to define precisely. Presumably, though, we do all have some familiarity with them, because they are usually introduced in secondary school, (not through definition but through illustrations with examples; definition is not the only way of creating concepts.)

Pythagoras' Thereom: *Given a right-angled triangle, then the area of the square on the longest side is the sum of the areas of the squares on the other two sides.*

Let A be the largest square and let B and C be the other two squares, as indicated in the left-hand part of the next diagram. The theorem says that

$$|A| = |B| + |C| \, .$$

We are using vertical bars to indicate quantities, in this case, areas. This notation may be unfamiliar to some of us but, as will transpire below, we do have good reason for distinguishing between an object X and its magnitude $|X|$.



A version of Euclid's second proof of the theorem: By considering angles, we see that the given triangle a can be cut up into two triangles b and c with the same shape as a, as indicated above. Plainly, $|A|/|a| = |B|/|b| = |C|/|c|$. But $|a| = |b| + |c|$, hence $|A| = |B| + |C|$. $\square$

But Euclid's version of the argument is more complicated, expressed in terms of lengths. As we shall see, formulation in terms of lengths sails into troubled waters. The theorem says that, given a right-angled triangle with longest edge α and other edges β and γ, then

$$|\alpha|^2 = |\beta|^2 + |\gamma|^2 \, .$$

Again, we are using vertical bars to indicate magnitudes, this time, lengths. Euclid cuts up the given triangle as before, then observes that, labelling the edges as depicted, we have $|\delta|/|\beta| = |\beta|/|\alpha|$, hence $|\beta|^2/|\alpha|^2 = |\delta|/|\alpha|$. Similarly, $|\gamma|^2/|\alpha|^2 = |\epsilon|/|\alpha|$. The conclusion of the theorem now follows because $|\delta| + |\epsilon| = |\alpha|$.

But now suppose the given triangle is equilateral, in other words, $|\beta| = |\gamma|$. Then some trouble arises with a number that crops up in the latest version of the proof. The theorem says that the number $\rho = |\alpha|/|\beta| = |\beta|/|\delta|$ satisfies

$$\rho^2 = 2 \, .$$

By straightforward calculation, it can be shown that, as an approximation to 2 decimal places,

$$\rho \approx 1.41 \qquad \text{in other words} \qquad \rho \approx 141/100\ .$$

By further calculation, it can be shown that, to 7 decimal places,

$$\rho \approx 1.41421436 \qquad \text{in other words} \qquad \rho \approx 14142136/10000000\ .$$

Unfortunately, using decimal notation — our conventional notation for expressing measured numerical quantities — we cannot express ρ exactly. In fact, by one of the most famous little arguments in the whole of mathematics, ρ cannot be expressed in the form n/m where n and m are positive integers. **If something cannot be expressed, then how can we validly use it in a deductive argument, and how can we even be sure that it exists?**

The following definitions are taken from *Book 5*.

Definition 4: *Two things with magnitude X and Y are said to have a ratio $X : Y$ provided some multiple of X exceeds Y and some multiple of Y exceeds X.*

Definition 7: *A ratio $X : Y$ exceeds a ratio $X' : Y'$, if and only if there exist positive integers n and m such that m copies of X exceed n copies of Y while m copies of X' do not exceed n copies of Y'. In that case, we write $X : Y > X' : Y'$.*

From our modern perspective, we might think of ratios as positive real numbers, identifying $X : Y$ with $|X|/|Y|$. But the Greeks did not treat them that way. They never added ratios together. However, the above definitions do resolve the trouble. Definition 4 allow us to make sense of ρ as the ratio $\rho = \alpha : \beta$. Definition 7 allows us to say that

$$14142136 : 10000000 > \rho > 141 : 100\ .$$

These ideas, attributed to Eudoxus, were resurrected by Richard Dedekind during the 1850s, date of publication 1872. The essential content of Definition 4 is nowadays called the *Archimedian Property of the Reals*, that of Definition 7, the *Density of the Rationals in the Reals*. They form part of the foundation for an area of mathematics called real analysis.

A parallel study of the development and reception of the material in the 1870s and 80s would be relevent but too long a digression. Let me just say that, in view of our comparatively detailed historical data pertaining to those two decades, I find it hard to imagine Eudoxus' definitions emerging merely as a response to niggling queries about proofs of fundamental theorems. What seems far more plausible, to me, would be that the concept of a ratio first arose in much more sophisticated contexts, the reassessment of fundamental pedagogical material coming later.

A speculative chronology

The traditional narrative is that Eudoxus extended an old theory of ratios which involved only ratios of positive integers. Such ratios would correspond, in modern terms, to the positive rational numbers. But the earliest source for that narrative appears to be Proclus, writing more than 800 years later. He had no basis for judgement, because he was writing during a dark era when the art of mathematical definition was fast asleep. His story does not ring true, in view of our section on reversal of chronology.

A natural surmise, from the discussion in the previous section, might be that the theory of ratios was developed to perfect the proofs of some fundamental theorems, such as Pythagoras' Theorem. Again, that surmise does not ring true, for the same reason.

One useful clue is in the Greek word for ratio: *logos*. Of course, that word has many meanings; we encountered it already above. My guess is that, as the name of a mathematical object, it simply means *word, expression*. The negative adjective *alogon* appears in the title of a lost book of Democritus. I suggest that the noun *logos* was subsequently selected to emphasize that the things in question, though difficult to express, nevertheless can be expressed, and can be used validly in deductive arguments.

The proposed chronology is as follows.

1: In the late 5th and early 4th centuries BC, the troubles indicated above led to serious difficulties in the very substantial mathematics that was being done by Archytas and Eudoxus; possibly in connection with a technique, called the *method of exhaustion*, that is nowadays associated with limits and integration. The troubles were addressed using the notion of ratio.
2: Over the course of many decades, the notion of a ratio was further clarified. In parallel to that, reasoning by definition and deduction spread to other parts of mathematics. It was also carried beyond mathematics by philosophers such as Plato and, eventually, Aristotle.
3: The logic initiated by Plato and fully established by Aristotle fed back into mathematics, heavily influencing Euclid's *Elements*.

To defend that proposed chronology, a cumulative weight of several lines of evidence would be needed. Let us supply just one line of evidence, from Plato's *Meno*.

First hour of the three-hour dialogue: Socrates objects to all of Meno's attempts to characterize virtue. He requests a general definition of virtue, not a list of kinds or a list of attributes.

Second hour: He gives a mathematics class to a slave-boy. On the basis of the responses from the boy, Socrates argues that mathematical insight arises from our memories of essential forms which our souls encountered in heaven before birth.

Third hour: With the theory of forms now reviewed, he guides some further attempts to characterize virtue. The project is largely unsuccessful. At the end, he confesses that the provisional conclusion 'Virtue is a gift from the gods" remains uncertain, because of a failure to nail down exactly what virtue is, "We will only truly know about this, however, if and when we try to investigate what virtue itself is; an investigation that must preceed that of how it comes to be in people."

If Plato had been trying to demonstrate the power of reasoning by definition and deduction, he would surely have chosen an illustration where a definition is supplied, a conclusion deduced.

Possible Explanation: Plato's intended readers are already familiar with reasoning by definition and deduction. Moreover, the readers will be satisfied just with a formulation of the problem, because they already know that finding the right definition can be very difficult.

But a long first hour is spent indicating why ordinary rhetorical characterizations of a concept are unsatisfactory.

Possible Explanation: The intended readers are *not* familiar with definition and deduction in the context of ethics.

Much of the middle hour is expended on a discussion of mathematics. The mathematical content is essentially equivalent to a proof of Pythagoras' Theorem in the special case of an equilateral right-angled triangle; precisely the case where the trouble $\rho^2 = 2$ arises.

Speculation: Plato is attempting to transfer a mode of reasoning from mathematics, where its use is familiar, to ethics, where its use is new.