

Documentation for:  
MATH 110, Discrete Mathematics.  
Spring 2009. Laurence Barker, Bilkent University

Source file: collSpr09.110.tex, latest revision: 24 March 2008

Page 2: Midterm I exam paper.

Page 3: Solutions to Midterm II, common mistakes, mistakes of special distinction.

## MATH 110: DISCRETE MATHEMATICS. Midterm I, Spring 09

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript. Do take the question-sheet home with you and, if possible, remember to bring it to the next class. The use of telephones and other electronic devices is prohibited.

LJB, 12 March 2009, Bilkent University.

**1: 15%** For which integers  $n \geq 1$  is  $K_n$  a planar graph? (Recall that  $K_n$  is the graph with  $n$  vertices and  $n(n-1)/2$  edges. You may use standard results about the number of vertices and the number of edges of a planar graph.)

**2: (a) 5%** State Euler's Criterion for a connected graph to have an Euler trail. (You do not need to give a proof of this.)

**(b) 15%** Give a similar criterion for a connected graph to have a walk that uses each edge an odd number of times. Justify your answer using part (a).

**3: 20%** Let  $a, b, c$  be complex numbers with  $a \neq 0$ . Suppose that the quadratic equation  $at^2 + bt + c = 0$  has a repeated non-zero solution  $\alpha$ . Let  $x_1, x_2, \dots$  be an infinite sequence of complex numbers such that

$$ax_{n+2} + bx_{n+1} + cx_n = 0.$$

Show that there exist complex numbers  $A$  and  $B$  such that

$$x_n = (A + nB)\alpha^n.$$

(Warning: It is not enough just to show that the formula is a solution to the recurrence relation. You must use mathematical induction to prove that every solution has that form.)

**4: 20%** Let  $x_1, x_2, \dots$  be an infinite sequence of complex numbers such that  $x_1 = x_2 = 0$  and

$$x_{n+2} - 2ix_{n+1} - x_n = 1 - in.$$

Find a formula for  $x_n$ . (Hint: First try  $x_n = (n-1)/2$ ; but note that, unfortunately, this does not satisfy the initial conditions.)

**5:** Let  $G$  be a connected planar graph such that every vertex has the same degree. Suppose that  $G$  can be drawn on the plane in such a way that every face has four edges.

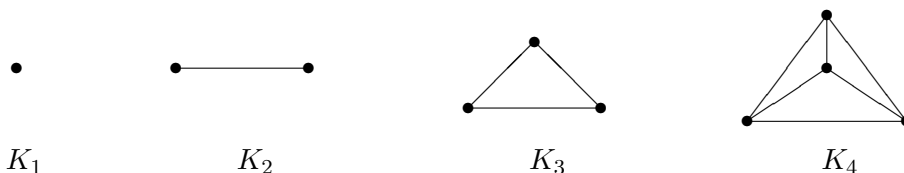
**(a) 5%** Show that  $2f = e$ .

**(b) 15%** Show that  $n = 4$  or  $n = 8$ , where  $n$  is the number of vertices.

**(c) 5%** Give an example of such a graph with  $n = 4$  and an example with  $n = 8$ .

## Solutions to Midterm 1.

**1:** The graph  $K_n$  is planar for  $n \leq 4$ . Indeed, the graphs  $K_1, K_2, K_3, K_4$  can be drawn as shown in the diagram.



Recall that, given a planar graph with  $n$  vertices and  $e$  edges, with  $e \geq 3$ , then  $e \leq 3n - 6$ . For  $K_5$ , we have  $n = 5$  and  $e = 10$ . If  $K_5$  is planar then  $10 \leq 3 \cdot 5 - 6 = 9$ , a fallacy. So  $K_5$  is non-planar. Finally, for  $n \geq 6$ , the graph  $K_n$  has  $K_5$  as a subgraph, so  $K_n$  cannot be planar.

**2: (a)** A connected graph has an Euler trail if and only if the number of vertices with odd degree is 0 or 2.

**(b)** Let us call such a walk an **odd walk**. A connected graph has an odd walk if and only if the number of vertices with odd degree is 0 or 2. *Proof:* First suppose there are precisely 0 or 2 vertices with odd degree. Then there is an Euler trail by part (a). Any Euler trail is an odd walk. Conversely, suppose that there is an odd walk from vertex  $s$  to vertex  $t$ . For any other vertex  $x$ , the number of times we enter  $x$  is equal to the number of times we exit  $x$ . So  $d(x)$  is even. If  $s = t$  then, again, the number of times we enter  $s$  is equal to the number of times we exit  $s$ , hence  $d(s)$  is even and the number of vertices with odd degree is even. On the other hand, if  $s \neq t$ , then we enter  $s$  one less time than we exit  $s$ , and we enter  $t$  one more time than we exit  $t$ . Hence  $d(s)$  and  $d(t)$  are odd and the number of vertices with odd degree is 2.

**3:** We argue by induction. Let  $A = x_0$  and  $B = x_1/\alpha - A$ . Base step: we have  $x_0 = A = (A + 0 \cdot B)\alpha^0$  and  $x_1 = (A + B)\alpha = (A + 1 \cdot B)\alpha^1$ . So the assertion holds for  $n = 0$  and  $n = 1$ . Induction Step: Fix  $n$  and suppose that  $x_n = (A + nB)\alpha^n$  and  $x_{n+1} = (A + (n+1)B)\alpha^{n+1}$ . Then

$$\begin{aligned} & a(A + (n+2)B)\alpha^{n+2} + bx_{n+1} + cx_n \\ &= a(A + (n+2)B)\alpha^{n+2} + b(A + (n+1)B)\alpha^{n+1} + c(A + nB)\alpha^n \\ &= (A + nB)\alpha^n(a\alpha^2 + b\alpha + c) + B\alpha^{n+1}(2a\alpha + b) = 0 = ax_{n+2} + bx_{n+1} + cx_n \end{aligned}$$

because  $a\alpha^2 + b\alpha + c = 0$  and  $\alpha = -b/2a$ . Canceling, we obtain  $x_{n+2} = (A + (n+2)B)\alpha^{n+2}$ , as required.

**4:** Letting  $y_n = (n-1)/2$ , it is easy to see that  $y_{n+2} - 2iy_{n+1} - y_n = 1 - in$ . Putting  $x_n = y_n + z_n$ , then  $z_n$  satisfies the homogeneous recurrence relation  $z_{n+2} - 2iz_{n+1} - z_n = 0$ . The auxiliary quadratic equation  $t^2 - 2it - 1 = 0$  has a repeated solution  $t = i$ , so  $z_n = (A + nB)i^n$  for some  $A$  and  $B$ . Thus  $x_n = (n-1)/2 + (A + nB)i^n$ .

We now solve for  $A$  and  $B$  using the initial conditions. We have  $0 = x_1 = (A + B)i$  and  $0 = x_2 = 1/2 + (A + 2B)i^2 = 1/2 - A - 2B$ . Therefore  $B = 1/2 = -A$ . In conclusion,

$$x_n = (n-1)/2 + (n/2 - 1/2)i^n = (n-1)(1 + i^n)/2.$$

**5: (a)** Consider the pairs  $(\epsilon, F)$  where  $\epsilon$  is an edge on a face  $F$ . Since each face has 4 edges, the number of such pairs is  $4f$ . But each edge belongs to 2 faces, so the number of such pairs is  $2e$ . Therefore  $2f = e$ .

(b) Since  $G$  is a connected planar graph,  $n - e + f = 2$ . By part (a),  $2n - e = 4$ . Letting  $d$  be the degree of the vertices, then  $2e = dn$ , hence  $4n - dn = 8$ , in other words,  $n = 8/(4 - d)$ . But  $n$  and  $d$  are integers with  $n \geq 0 \leq d$ , so  $d \in \{0, 2, 3\}$ . But, if  $d = 0$  then the graph has no edges, yet  $n = 2$ . This is impossible, because a graph with 2 vertices and no edges is not connected. So  $d = 2$  or  $d = 3$ , hence  $n = 4$  or  $n = 8$ .

(c) For  $n = 4$ , the vertices and edges of a square comprise such a graph. For  $n = 8$ , the vertices and edges of a cube comprise such a graph. (I omit the diagrams, which would take a long while to produce using my LaTeX word-processing program.)

### Comments on common mistakes in Midterm 1

**1:** One person disposed of the case  $n \geq 5$  using Kurotowsky's Theorem (not on the syllabus). But one ought to avoid using deep theorems to prove trivial little results, because that gives a false impression of difficulty. Furthermore, sometimes the trivial little result is used in the proof of the deep theorem. I have not seen a proof of Kurotowsky's theorem, but I have no doubt that the non-planarity of  $K_5$  is a necessary lemma. I did, however, award full marks in this case.

Others argued as follows:

*For  $K_n$ , we have  $e = n(n - 1)/2$ . If  $K_n$  is planar, then  $e \leq 3n - 6$ , hence  $(n - 3)(n - 4) \leq 0$ , so  $n = 3$  or  $n = 4$ . Therefore  $K_n$  is planar if and only if  $n = 3$  or  $n = 4$ .*

As a way of dealing with the case  $n \geq 5$ , this is very elegant. To complete the argument, one just has to deal with the case  $n \leq 4$ , and it would be enough to draw the four diagrams as in my solution above. However, as the argument stands, there are two mistakes.

One of the mistakes lies in the fact that the inequality  $e \leq 3n - 6$  is valid only when  $e \geq 3$ . (Recall that, in the proof of that inequality, presented in class, we made use of the fact that every face has at least three edges.) So the above argument applies only when  $n \geq 3$ . We must deal separately with the cases  $n = 1$  and  $n = 2$  (and, indeed, the graph  $K_n$  is planar in those two cases).

The other mistake — made by many people — lies in the fact that the criterion works only one way: Assuming that  $e \geq 3$ , then planar implies  $e \leq 3n - 6$ , but  $e \leq 3n - 6$  does not imply planar. The argument shows that  $K_3$  and  $K_4$  satisfy  $e \leq 3n - 6$ , but it does not show that  $K_3$  and  $K_4$  are planar. To confirm that  $K_4$  is planar, I think one ought to draw an appropriate diagram. To confirm that  $K_1, K_2, K_3$  are planar, one might as well again draw diagrams. (Or, alternatively, I think the reader would understand if one were just to point out that  $K_1, K_2, K_3$  are obviously planar. But this is still a situation where, at the very least, one ought to alert the reader to the fact that something is obvious, just to reassure the reader that the matter has been considered.)

I awarded 11/15 for the above solution with the wrong answer  $n \in \{3, 4\}$ . I awarded 13/15 for the above argument combined with the right answer  $n \leq 4$  yet without an explanation as to why  $K_n$  is planar for  $n \leq 4$ . Here are two interesting variants of of the argument.

One person gave the above argument and then tried to deal with the case  $n \leq 4$  as follows: "For  $n = 1, e = 0, f = 1$ , have  $n - e + f = 1 - 0 + 1 = 2$ , planar. For  $n = 2, e = 1, f = 1$ , have  $n - e + f = 2 - 1 + 1 = 2$ , planar. For  $n = 3, e = 3, f = 2$ , have  $n - e + f = 3 - 3 + 2 = 2$ , planar. For  $n = 4, e = 6, f = 4$ , have  $n - e + f = 4 - 6 + 4 = 2$ , planar." But the number of faces  $f$  is defined only for planar graphs. No diagrams were drawn, but perhaps the writer mentally visualized them and counted the numbers of faces for  $K_1, K_2, K_3, K_4$ . If so, then

the diagrams would already confirm that those four graphs are planar! Or perhaps the writer just solved for  $f$  using the equation  $n - e + f = 2$ . If so, then the calculations imply nothing. Mark: 13/15.

Another candidate wrote the following, with no words at all:

$$3n - 6 \geq e, n \geq 1, 3n - 6 \geq n(n - 1)/2, 6n - 12 \geq n^2 - n, 0 \geq n^2 - 7n + 12, \\ 0 \geq (n - 3)(n - 4), n = 3, n = 4, K_3, K_4.$$

What does any of that mean? Well, sure enough, if we already know the argument, then it is easy to guess what the writer is thinking. But the whole point of a proof is to explain something to someone who does not already know it. Does the inequality  $3n - 6 \geq e$  hold for all  $n$ , or just for some  $n$ , or is it an assumption? (Actually, it holds when  $K_n$  is planar and  $e \geq 3$ .) What does the writer mean by ending with the symbols  $K_3, K_4$ ? Does he or she mean to say that  $K_3$  and  $K_4$  are planar graphs, or that they are non-planar graphs, or that they are rabbits? (Actually, the calculations have merely shown that  $K_3$  and  $K_4$  satisfy the inequality  $3n - 6 \geq e$ .) Mark awarded: 5/15. *You must use words to explain what is going on.*

**2:** Language point: we may say that a graph *is* connected or that it *has* an Euler trail.

**(a)** Several people answered with something like: *A connected graph has an Euler trail if and only if exactly 2 of the vertices have odd degree. The graph has an Euler circuit if and only if all of the vertices have even degree. An Euler circuit is also an Euler trail.* This does not make sense. If all the vertices have even degree, is there an Euler trail or not? Answer: yes, there is an Euler trail.

The mistake might be related to the fact that, in class, I expressed part of a theorem as *The graph has an Euler trail but no Euler circuit if and only if exactly 2 of the vertices have odd degree.* Every rabbit is a mammal, but some mammals are not rabbits. So, given a mammal, it might be a rabbit, or it might not be a rabbit. (Are you with me so far?) If every rabbit has ten legs and if every other mammal has eleven legs, then it follows that every mammal has ten or eleven legs.

Another daft answer: “For a connected graph to have an Euler trail, we must have 2 vertices with odd degree and each edge is visited only once.” This is a deranged mixture of Euler’s *criterion* for the existence of an Euler trail (a condition concerning a graph) and the *definition* of an Euler trail (a condition concerning a trail). A corrected version: *For a connected graph to have an Euler trail, we must have exactly 0 or 2 vertices with odd degree.* Tip: do not add extra spurious information in the hope of picking up floating marks. That would just make things more difficult for a reader, so the examiner is likely to subtract marks. (In this case, I subtracted one mark for omitting the 0 vertices of odd degree, and I subtracted another mark for the confusing extra information.)

**(b)** Many people responded with one or two paragraphs of amorphous discussion, but without giving an answer. *A connected graph has such a walk if and only if ... what?* I do not award many marks to people who just make correct relevant statements without arriving at any conclusion. So be brave: commit yourself to a definite answer, and then try to prove it.

**3:** Rather disappointing. From the scripts, anyone with scant experience of undergraduate teaching might surmise that many of the students have no grasp at all of the reasoning behind induction arguments! Actually, my guess is that most of the people in the class do understand the reasoning in the context of simple exercises, but many get confused in more complicated

situations. I shall try to remember to include at least one induction argument in Midterm II. Many scripts offered solutions with the following form:

$$\text{Base step: } a(A + 2B)\alpha^2 + b(A + B)\alpha + cA = \dots = 0. \text{ Induction step: } \\ a(A + (n + 2)B)\alpha^{n+2} + b(A + (n + 1)B)\alpha^{n+1} + C(A + nB)\alpha^n = \dots = 0.$$

Even when the manipulations were entirely correct (including the tricky part where one makes use of the equation  $\alpha = -b/2a$ ), I awarded no marks for offerings which had that form. (I do not think I am being harsh, because I issued a warning about this matter in the question.) But what is wrong with that attempted solution?

Well, first of all, just prefixing a calculation with “Base step: blah blah. Induction Step” does not magically turn the calculation into an argument by induction. Sure enough, the manipulation  $a(A + 2B)\alpha^2 + b(A + B)\alpha + cA = \dots = 0$  does actually prove something. It proves that if  $x_n = (A + nB)\alpha^n$  then  $ax_{n+2} + bx_{n+1} + cx_n = 0$ . But the question did not ask for a proof of that fact. It asked for a proof that if  $ax_{n+2} + bx_{n+1} + cx_n = 0$  then  $x_n = (A + nB)\alpha^n$  for some  $A$  and  $B$ .

The crux of the matter is to find the right inductive assumption. In the induction step, one shows that if we *assume* the formula for  $x_n$  and  $x_{n+1}$ , then we can *deduce* the formula for  $x_{n+2}$ . (The assumption does need to be specified clearly, because some people claimed, wrongly, that if we assume the formula just for  $x_{n-1}$  then we can deduce the formula for  $x_n$ .) As soon as the assumption in the induction step has been specified, the reader does not really need any more help with that step; any idiot can then carry out the manipulations. Indeed, for a more advanced reader, one could deal with the induction step just by saying: *Assuming the stated formula for  $x_n$  and  $x_{n+1}$ , then a straightforward manipulation using the equality  $\alpha = -b/2a$  yields the stated formula for  $x_{n+2}$ .* In the base step, too, it is necessary to give the reader some help. We have to explain that some particular values of  $A$  and  $B$  are to be selected, namely  $A = x_0$  and  $B = x_1/\alpha - A$ , whereupon the stated formula holds for  $x_0$  and  $x_1$ .

This ought to have been an easy question, since it was bookwork. I already presented the argument in class, and also in the internet notes `discrete2.pdf`.

**5 (a):** Many people offered arguments such as the following one: “(a)  $\bar{c} = 4$  and  $\bar{c}f = 2e$  so  $4f = 2e$ , so  $2f = e$ .” I admit that the requirements of the question might not have been sufficiently clear. (It is difficult to phrase bookwork questions for first-years in such a way that they know what can be assumed and what has to be proved.) The equations  $2f = e$  and  $\bar{c}f = 2e$  both occurred in the middle of arguments presented in class. But neither of them are standard lemmas. The question demanded a *proof* that  $2f = e$ , not just a quotation of the more general formula  $\bar{c}f = 2e$ . My scheme was to generously award 2/5 for the above, and 3/5 to those who explained that  $\bar{c}$  is the number of edges per face (or, more generally, the average number of edges per face). I awarded 5/5 to anyone who mentioned the key idea of the proof, which is to note that each edge is associated with two faces. (Only two people gave a really lucid counting-by-pairs argument).

**Olympiad results.** Or: How come we survived and the Neanderthals went extinct?

*Bronze Medal:* **5:** “(a)  $c = 4$  and  $cf = 2e$  so  $4f = 2e$ , so  $2f = e$ . (b) If  $G$  is planar then  $n - e + f = 2$ . For  $n = 4$  we have  $4 - 2f + f = 2$ , so  $f = 2$  and  $e = 4$  so  $4 - 4 + 2 = 2$ . So  $n = 4$ ,  $e = 4$ ,  $f = 2$  satisfy the condition. For  $n = 8$  we have  $8 - 2f + f = 2$  so  $f = 6$  and  $e = 12$  so  $8 - 12 + 6 = 2$ . So  $n = 6$ ,  $e = 12$ ,  $f = 6$  satisfy the condition.” (For readability, I have edited the English and I have inserted some words.)

Part (a), here, is tolerable; see a comment above. For part (b), the whole aim is to show that  $n$  cannot be anything other than 4 or 8. It is no use assuming the conclusion, then solving some equations, and then noting — how amazing! — that the solutions still satisfy the equations even after the solutions have been written down on paper.

*Silver medal: 5:* “ $n - e + f = 2$ . Suppose that our graph is a square. In our square, every vertex has the same degree,  $\deg(a) = \deg(b) = \deg(c) = \deg(d) = 2$ . Our faces  $f_1$  and  $f_2$  have four edges. **(a)**  $f = 2, e = 4$ , so  $2f = e, 2 \cdot 2 = 4, \checkmark$ . **(b)**  $n = 4, n - e + f = 2, f = 6, e = 12$ .” (The script has a diagram of a square with the vertices labeled  $a, b, c, d$  and the inside and outside faces labeled  $f_1$  and  $f_2$ . No doubt the final two equalities arise from some fragment of insight into the case  $n = 8$ .)

The particular does not imply the general, not even when the particular has been described in loving detail. My pet rabbit is a mammal. My pet rabbit is very tame, and it eats carrots from out of my hand without ever nibbling my fingers. Therefore all mammals are tame. In an earlier epoch, the name for people who reasoned this way was: sabre-tooth tiger food.

*Gold Medal: 3:* “In this question, let us assume that the given question is our initial condition,  $x_n = (A + nB)\alpha^n$ .”