A new canonical induction formula for p-permutation modules

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1 October 2018

Abstract

Applying Robert Boltje's theory of canonical induction, we give a restriction-preserving formula expressing any p-permutation module as a $\mathbb{Z}[1/p]$-linear combination of modules induced and inflated from projective modules associated with subquotient groups. The underlying constructions include, for any given finite group, a ring with a $\mathbb{Z}$-basis indexed by conjugacy classes of triples (U, K, E) where U is a subgroup, K is a p'-residue-free normal subgroup of U and E is an indecomposable projective module of the group algebra of U/K.

2010 *Mathematics Subject Classification:* 20C20.

1 Introduction

We shall be applying Boltje's theory of canonical induction [2] to the ring of p-permutation modules. Of course, p is a prime. We shall be considering p-permutation modules for finite groups over an algebraically closed field $\mathbb{F}$ of characteristic p. A review of the theory of p-permutation modules can be found in Bouc–Thévenaz [5, Section 2].

A canonical induction formula for p-permutation modules was given by Boltje [3, Section 4] and shown to be $\mathbb{Z}$-integral. It expresses any p-permutation module, up to isomorphism, as a $\mathbb{Z}$-linear combination of modules induced from a special kind of p-permutation module, namely, the 1-dimensional modules.

We shall be inducing from another special kind of p-permutation module. Let G be a finite group. We understand all $\mathbb{F}G$-modules to be finite-dimensional. An indecomposable $\mathbb{F}G$-module M is said to be **exprojective** provided the following equivalent conditions hold up to isomorphism: there exists a normal subgroup $K \trianglelefteq G$ such that M is inflated from a projective $\mathbb{F}G/K$-module; there exists $K \trianglelefteq G$ such that M is a direct summand of the permutation $\mathbb{F}G$-module $\mathbb{F}G/K$; every vertex of M acts trivially on M; some vertex of M acts trivially on

M. Generally, an $\mathbb{F}G$-module X is called **exprojective** provided every indecomposable direct summand of X is exprojective.

The exprojective modules do already play a special role in the theory of p-permutation modules. Indeed, the parametrization of the indecomposable p-permutation modules, recalled in Section 2, characterizes any indecomposable p-permutation module as a particular direct summand of a module induced from an exprojective module.

We shall give a $\mathbb{Z}[1/p]$-integral canonical induction formula, expressing any p-permutation $\mathbb{F}G$-module, up to isomorphism, as a $\mathbb{Z}[1/p]$-linear combination of modules induced from exprojective modules. More precisely, we shall be working with the Grothendieck ring for p-permutation modules $T(G)$ and we shall be introducing another commutative ring $\mathcal{T}(G)$ which, roughly speaking, has a free $\mathbb{Z}$-basis consisting of lifts of induced modules of indecomposable exprojective modules. Letting $\mathbb{K}$ be a field of characteristic zero that is sufficiently large for our purposes, we shall consider a ring epimorphism $\mathrm{lin}_G : \mathcal{T}(G) \to T(G)$ and its $\mathbb{K}$-linear extension $\mathrm{lin}_G : \mathbb{K}\mathcal{T}(G) \to \mathbb{K}T(G)$. The latter is split by a $\mathbb{K}$-linear map $\mathrm{can}_G : \mathbb{K}T(G) \to \mathbb{K}\mathcal{T}(G)$ which, as we shall show, restricts to a $\mathbb{Z}[1/p]$-linear map $\mathrm{can}_G : \mathbb{Z}[1/p]T(G) \to \mathbb{Z}[1/p]\mathcal{T}(G)$.

To motivate further study of the algebras $\mathbb{Z}[1/p]\mathcal{T}(G)$ and $\mathbb{K}\mathcal{T}(G)$, we mention that, notwithstanding the formulas for the primitive idempotents of $\mathbb{K}T(G)$ in Boltje [4, 3.6], Bouc–Thévenaz [5, 4.12] and [1], the relationship between those idempotents and the basis $\{[M^G_{P,E}] : (P, E) \in_G \mathcal{P}(E)\}$ remains mysterious. In Section 4, we shall prove that $\mathbb{K}\mathcal{T}(G)$ is $\mathbb{K}$-semisimple as well as commutative, in other words, the primitive idempotents of $\mathbb{K}\mathcal{T}(G)$ comprise a basis for $\mathbb{K}\mathcal{T}(G)$. We shall also describe how, via lin_G, each primitive idempotent of $\mathbb{K}T(G)$ lifts to a primitive idempotent of $\mathbb{K}\mathcal{T}(G)$.

2 Exprojective modules

We shall establish some general properties of exprojective modules.

Given $H \leq G$, we write ${}_G\mathrm{Ind}_H$ and ${}_H\mathrm{Res}_G$ to denote the induction and restriction functors between $\mathbb{F}G$-modules and $\mathbb{F}H$-modules. When $H \trianglelefteq G$, we write ${}_G\mathrm{Inf}_{G/H}$ to denote the inflation functor to $\mathbb{F}G$-modules from $\mathbb{F}G/H$-modules. Given a finite group L and an understood isomorphism $L \to G$, we write ${}_L\mathrm{Iso}_G$ to denote the isogation functor to $\mathbb{F}L$-modules from $\mathbb{F}G$-modules, we mean to say, ${}_L\mathrm{Iso}_G(X)$ is the $\mathbb{F}L$-module obtained from an $\mathbb{F}G$-module X by transport of structure via the understood isomorphism.

Let us classify the exprojective $\mathbb{F}G$-modules up to isomorphism. We say that G is **p'-residue-free** provided $G = O^{p'}(G)$, equivalently, G is generated by the Sylow p-subgroups of G. Let $\mathcal{Q}(G)$ denote the set of pairs (K, F), where K is a p'-residue-free normal subgroup of G and F is an indecomposable projective $\mathbb{F}G/K$-module, two such pairs (K, F) and (K', F') being deemed the same provided $K = K'$ and $F \cong F'$. We define an indecomposable exprojective $\mathbb{F}G$-module $M^{K,F}_G = {}_G\mathrm{Inf}_{G/K}(F)$. By considering vertices, we obtain the following result.

Proposition 2.1. *The condition $M \cong M^{K,F}_G$ characterizes a bijective correspondence between:*
(a) *the isomorphism classes of indecomposable exprojective $\mathbb{F}G$-modules M,*
(b) *the elements (K, F) of $\mathcal{Q}(G)$.*

In particular, for a p-subgroup P of G, the condition $E \cong {}_{N_G(P)}\mathrm{Inf}_{N_G(P)/P}(\overline{E})$ characterizes a bijective correspondence between, up to isomorphism, the indecomposable exprojective $\mathbb{F}N_G(P)$-modules E with vertex P and the indecomposable projective $\mathbb{F}N_G(P)/P$-modules $\overline{E}$. It follows that the well-known classification of the isomorphism classes of indecomposable p-permutation $\mathbb{F}G$-modules, as in Bouc–Thévenaz [5, 2.9] for instance, can be expressed as in the next result. Let $\mathcal{P}(G)$ denote the set of pairs (P,E) where P is a p-subgroup of G and E is an exprojective $\mathbb{F}N_G(P)$-module with vertex P, two such pairs (P,E) and (P',E') being deemed the same provided $P=P'$ and $E \cong E'$. We make $\mathcal{P}(G)$ become a G-set via the actions on the coordinates. We define $M^G_{P,E}$ to be the indecomposable p-permutation $\mathbb{F}G$-module with vertex P in Green correspondence with E.

Theorem 2.2. *The condition $M \cong M^G_{P,E}$ characterizes a bijective correspondence between:*
(a) *the isomorphism classes of indecomposable p-permutation $\mathbb{F}G$-modules M,*
(b) *the G-conjugacy classes of elements $(P,E) \in \mathcal{P}(G)$.*

We now give a necessary and sufficient condition for $M^G_{P,E}$ to be exprojective.

Proposition 2.3. *Let $(P,E) \in \mathcal{P}(G)$. Let K be the normal closure of P in G. Then $M^G_{P,E}$ is exprojective if and only if $N_K(P)$ acts trivially on E. In that case, K is p'-residue-free, P is a Sylow p-subgroup of K, we have $G = N_G(P)K$, the inclusion $N_G(P) \hookrightarrow G$ induces an isomorphism $N_G(P)/P \cong G/K$, and $M^G_{P,E} \cong M^{K,F}_G$, where F is the indecomposable projective $\mathbb{F}G/K$-module determined, up to isomorphism, by the condition $E \cong {}_{N_G(P)}\mathrm{Inf}_{N_G(P)/P}\mathrm{Iso}_{G/K}(F)$.*

Proof. Write $M = M^G_{P,E}$. If M is exprojective then K acts trivially on M and, perforce, $N_K(P)$ acts trivially on E.

Conversely, suppose $N_K(P)$ acts trivially on E. Then P, being a vertex of E, must be a Sylow p-subgroup of $N_K(P)$. Hence, P is a Sylow p-subgroup of K. By a Frattini argument, $G = N_G(P)K$ and we have an isomorphism $N_G(P)/P \cong G/K$ as specified. Let $X = {}_G\mathrm{Ind}_{N_G(P)}(E)$. The assumption on E implies that X has well-defined $\mathbb{F}$-submodules

$$Y = \left\{ \sum\nolimits_k k \otimes_{N_G(P)} x : x \in E \right\} , \qquad Y' = \left\{ \sum\nolimits_k k \otimes_{N_G(P)} x_k : x_k \in E, \sum\nolimits_k x_k = 0 \right\}$$

summed over a left transversal $kN_K(P) \subseteq K$. Making use of the well-definedness, an easy manipulation shows that the action of $N_G(P)$ on X stabilizes Y and Y'. Similarly, K stabilizes Y and Y'. So Y and Y' are $\mathbb{F}G$-submodules of X. Since $|K : N_K(P)|$ is coprime to p, we have $Y \cap Y' = 0$. Since $|K : N_K(P)| = |G : N_G(P)|$, a consideration of dimensions yields $X = Y \oplus Y'$.

Fix a left transversal $\mathcal{L}$ for $N_K(P)$ in K. For $g \in N_G(P)$ and $\ell \in \mathcal{L}$, we can write ${}^g\ell = \ell_g h_g$ with $\ell_g \in \mathcal{L}$ and $h_g \in N_K(P)$. By the assumption on E again, $h_g x = x$ for all $x \in E$. So

$$g \sum\nolimits_\ell \ell \otimes x = \sum\nolimits_\ell {}^g\ell \otimes gx = \sum\nolimits_\ell \ell_g \otimes gx = \sum\nolimits_\ell \ell \otimes gx$$

summed over $\ell \in \mathcal{L}$. We have shown that ${}_{N_G(P)}\mathrm{Res}_G(Y) \cong E$. A similar argument involving a sum over $\mathcal{L}$ shows that K acts trivially on Y. Therefore, $Y \cong M^{K,F}_G$. On the other hand, Y is indecomposable with vertex P and, by the Green correspondence, $Y \cong M^G_{P,E}$. □

We shall be making use of the following closure property.

Proposition 2.4. *Given exprojective $\mathbb{F}G$-modules X and Y, then the $\mathbb{F}G$-module $X \otimes_\mathbb{F} Y$ is exprojective.*

Proof. We may assume that X and Y are indecomposable. Then X and Y are, respectively, direct summands of permutation $\mathbb{F}G$-modules having the form $\mathbb{F}G/K$ and $\mathbb{F}G/L$ where $K \trianglelefteq G \trianglerighteq L$. By Mackey decomposition and the Krull–Schmidt Theorem, every indecomposable direct summand of $X \otimes Y$ is a direct summand of $\mathbb{F}G/(K \cap L)$. □

3 A canonical induction formula

Throughout, we let $\mathfrak{K}$ be a class of finite groups that is closed under taking subgroups. We shall understand that $G \in \mathfrak{K}$. In clarification of a hypothesis imposed in Section 1, we define $\mathbb{K}$ to be a field of characteristic zero that splits for all the groups in $\mathfrak{K}$. We shall abuse notation, neglecting to use distinct expressions to distinguish between a linear map and its extension to a larger coefficient ring.

Specializing some general theory in Boltje [2], we shall introduce a commutative ring $\mathcal{T}(G)$ and a ring epimorphism $\mathrm{lin}_G : \mathcal{T}(G) \to T(G)$. We shall show that the $\mathbb{Z}[1/p]$-linear extension $\mathrm{lin}_G : \mathbb{Z}[1/p]\mathcal{T}(G) \to \mathbb{Z}[1/p]T(G)$ has a splitting $\mathrm{can}_G : \mathbb{Z}[1/p]T(G) \to \mathbb{Z}[1/p]\mathcal{T}(G)$. As we shall see, can_G is the unique splitting that commutes with restriction and isogation.

To be clear about the definition of $T(G)$, the Grothendieck ring of the category of p-permutation $\mathbb{F}G$-modules, we mention that the split short exact sequences are the distinguished sequences determining the relations on $T(G)$. The multiplication on $T(G)$ is given by tensor product over $\mathbb{F}$. Given a p-permutation $\mathbb{F}G$-module X, we write $[X]$ to denote the isomorphism class of X. We understand that $[X] \in T(G)$. By Theorem 2.2,

$$T(G) = \bigoplus_{(P,E) \in_G \mathcal{P}(G)} \mathbb{Z}[M^G_{P,E}]$$

as a direct sum of regular $\mathbb{Z}$-modules, the notation indicating that the index runs over representatives of G-orbits. Let $T^{\mathrm{ex}}(G)$ denote the $\mathbb{Z}$-submodule of $T(G)$ spanned by the isomorphism classes of exprojective $\mathbb{F}G$-modules. By Proposition 2.4, $T^{\mathrm{ex}}(G)$ is a subring of $T(G)$. By Proposition 2.1

$$T^{\mathrm{ex}}(G) = \bigoplus_{(K,F) \in_G \mathcal{Q}(G)} \mathbb{Z}[M^{K,F}_G] \, .$$

For $H \leq G$, the induction and restriction functors ${}_G\mathrm{Ind}_H$ and ${}_H\mathrm{Res}_G$ give rise to induction and restriction maps ${}_G\mathrm{ind}_H$ and ${}_H\mathrm{res}_G$ between $T(H)$ and $T(G)$. Similarly, given $L \in \mathfrak{K}$ and an isomorphism $\theta : L \to G$, we have an evident isogation map ${}_L\mathrm{iso}^\theta_G : T(L) \leftarrow T(G)$. In particular, given $g \in G$, we have an evident conjugation map ${}_{{}^gH}\mathrm{con}^g_H$. Boltje noted that, when $\mathfrak{K}$ is the set of subgroups of a given fixed finite group, T is a Green functor in the sense of [2, 1.1c]. For arbitrary $\mathfrak{K}$, a class of admitted isogations must be understood, and the isogations and inclusions between groups in $\mathfrak{K}$ must satisfy the axioms of a category. Granted that, then T is still a Green functor in an evident sense whereby the conjugations replaced by isogations.

Following a construction in [2, 2.2], adaptation to the case of arbitrary $\mathfrak{K}$ being straightforward, we form the G-cofixed quotient $\mathbb{Z}$-module

$$\mathcal{T}(G) = \Big(\bigoplus_{U \leq G} T^{\mathrm{ex}}(U) \Big)_G$$

where G acts on the direct sum via the conjugation maps ${}_{{}^gU}\mathrm{con}^g_U$. Harnessing the Green functor structure of T, the restriction functor structure of T^{ex} and noting that $T^{\mathrm{ex}}(G)$ is a subring of $T(G)$, we make $\mathcal{T}$ become a Green functor much as in [2, 2.2], with the evident isogation maps. In particular, $\mathcal{T}(G)$ becomes a ring, commutative because $T(G)$ is commutative. Given $x_U \in T^{\mathrm{ex}}(U)$, we write $[U, x_U]_G$ to denote the image of x_U in $\mathcal{T}(G)$. Any $x \in \mathcal{T}(G)$ can be expressed in the form

$$x = \sum_{U \leq_G G} [U, x_U]_G$$

where the notation indicates that the index runs over representatives of the G-conjugacy classes of subgroups of G. Note that x determines $[U, x_U]$ and x_G but not, in general, x_U. Let $\mathcal{R}(G)$ be the G-set of pairs (U, K, F) where $U \leq G$ and $(K, F) \in \mathcal{Q}(U)$. We have

$$\mathcal{T}(G) = \bigoplus_{U \leq_G G, (K,F) \in_{N_G(U)} \mathcal{Q}(U)} \mathbb{Z}[U, [M_U^{K,F}]] = \bigoplus_{(U,K,F) \in_G \mathcal{R}(G)} \mathbb{Z}[U, [M_U^{K,F}]] \; .$$

We define a $\mathbb{Z}$-linear map $\mathrm{lin}_G : \mathcal{T}(G) \to T(G)$ such that $\mathrm{lin}_G[U, x_U] = {}_G\mathrm{ind}_U(x_U)$. As noted in [2, 3.1], the family $(\mathrm{lin}_G : G \in \mathfrak{K})$ is a morphism of Green functors $\mathrm{lin} : \mathcal{T} \to T$. In particular, the map $\mathrm{lin}_G : \mathcal{T}(G) \to T(G)$ is a ring homomorphism. Extending to coefficients in $\mathbb{K}$, we obtain an algebra map

$$\mathrm{lin}_G \; : \; \mathbb{K}\mathcal{T}(G) \to \mathbb{K}T(G) \; .$$

Let $\pi_G : T(G) \to T^{\mathrm{ex}}(G)$ be the $\mathbb{Z}$-linear epimorphism such that π_G acts as the identity on $T^{\mathrm{ex}}(G)$ and π_G annihilates the isomorphism class of every indecomposable non-exprojective p-permutation $\mathbb{F}G$-module. By $\mathbb{K}$-linear extension again, we obtain a $\mathbb{K}$-linear epimorphism $\pi_G : \mathbb{K}T(G) \to \mathbb{K}T^{\mathrm{ex}}(G)$. After [2, 5.3a, 6.1a], we define a $\mathbb{K}$-linear map

$$\mathrm{can}_G \; : \; \mathbb{K}T(G) \to \mathbb{K}\mathcal{T}(G) \; , \; \xi \mapsto \frac{1}{|G|} \sum_{U,V \leq G} |U| \, \mathrm{m\ddot{o}b}(U,V) [U, {}_U\mathrm{res}_V(\pi_V({}_V\mathrm{res}_G(\xi)))]_G$$

where möb() denotes the Möbius function on the poset of subgroups of G.

Theorem 3.1. *Consider the* $\mathbb{K}$*-linear map* can_G*.*
(1) *We have* $\mathrm{lin}_G \circ \mathrm{can}_G = \mathrm{id}_{\mathbb{K}T(G)}$*.*
(2) *For all* $H \leq G$*, we have* ${}_H\mathrm{res}_G \circ \mathrm{can}_G = \mathrm{can}_H \circ {}_H\mathrm{res}_G$*.*
(3) *For all* $L \in \mathfrak{K}$ *and isomorphisms* $\theta : L \leftarrow G$*, we have* ${}_L\mathrm{iso}_G^\theta \circ \mathrm{can}_G = \mathrm{can}_L \circ {}_L\mathrm{iso}_G^\theta$*.*
(4) $\mathrm{can}_G[X] = [X]$ *for all exprojective* $\mathbb{F}G$*-modules* X*.*
Those four properties, taken together for all $G \in \mathfrak{K}$*, determine the maps* can_G*.*

Proof. In view of the discussion above, this follows from the proof of [2, 5.3a]. □

Parts (2) and (3) of the theorem can be interpreted as saying that $\mathrm{can}_* : T \to \mathcal{T}$ is a morphism of restriction functors. It is not hard to check that, when $\mathfrak{K}$ is closed under the taking of quotient groups, the functors T, T^{ex}, $\mathcal{T}$ can be equipped with inflation maps, and the morphisms lin_* and can_* are compatible with inflation.

The latest theorem immediately yields the following corollary.

Corollary 3.2. *Given a* p*-permutation* $\mathbb{F}G$*-module* X*, then*

$$[X] = \frac{1}{|G|} \sum_{U,V \leq G} |U| \, \mathrm{m\ddot{o}b}(U,V) \, {}_G\mathrm{ind}_U \mathrm{res}_V(\pi_V({}_V\mathrm{res}_G[X])) \; .$$

Given p-permutation $\mathbb{F}G$-modules M and X, with M indecomposable, we write $m_G(M, X)$ to denote the multiplicity of M as a direct summand of X. We write $\pi_G(X)$ to denote the direct summand of X, well-defined up to isomorphism, such that $[\pi_G(X)] = \pi_G[X]$.

Lemma 3.3. *Let* $\mathfrak{p}$ *be a set of primes. Suppose that, for all* $V \in \mathfrak{K}$*, all* p*-permutation* $\mathbb{F}V$*-modules* Y*, all* $U \lhd V$ *such that* $|V : U| \in \mathfrak{p}$ *and all* V*-fixed elements* $(K, F) \in \mathcal{Q}(U)$*, we have*

$$m_U(M_U^{K,F}, \pi_U({}_U\mathrm{Res}_V(Y))) = \sum_{(J,E) \in \mathcal{Q}(V)} m_V(M_U^{K,F}, M_V^{J,E}) \, m_V(M_V^{J,E}, \pi_V(Y)) \; .$$

Then, for all $G \in \mathfrak{K}$*, we have* $|G|_{\mathfrak{p}'} \, \mathrm{can}_G[Y] \in \mathcal{T}(G)$*, where* $|G|_{\mathfrak{p}'}$ *denotes the* $\mathfrak{p}'$*-part of* $|G|$*.*

Proof. This is a special case of [2, 9.4]. Indeed, an easy inductive argument justifies our imposition of the condition $|V : U| \in \mathfrak{p}$ in place of the weaker condition that V/U is a cyclic $\mathfrak{p}$-group. □

We can now prove the $\mathbb{Z}[1/p]$-integrality of can_G.

Theorem 3.4. *The $\mathbb{K}$-linear map can_G restricts to a $\mathbb{Z}[1/p]$-linear map $\mathbb{Z}[1/p]T(G) \to \mathbb{Z}[1/p]\mathcal{T}(G)$.*

Proof. Let $\mathfrak{p}$ be the set of primes distinct from p. Let V, Y, U, K, F be as in the latest lemma. We must obtain the equality in the lemma. We may assume that Y is indecomposable. If Y is exprojective, then $\pi_U({}_U\mathrm{Res}_V(Y)) \cong {}_U\mathrm{Res}_V(Y)$ and $\pi_V(Y) \cong X$, whence the required equality is clear. So we may assume that Y is non-exprojective. Then $\pi_V(Y)$ is the zero module. It suffices to show that $M_U^{K,F}$ is not a direct summand of ${}_U\mathrm{Res}_V(Y)$. For a contradiction, suppose otherwise. The hypothesis on $|V : U|$ implies that U contains the vertices of Y. So $Y \mid {}_V\mathrm{Ind}_U(X)$ for some indecomposable p-permutation $\mathbb{F}U$-module X. Bearing in mind that (K, F) is V-stable, a Mackey decomposition argument shows that $M_U^{K,F} \cong X$. The V-stability of (K, F) also implies that $K \lhd V$. So

$$Y \mid {}_V\mathrm{Ind}_U\mathrm{Inf}_{U/K}(F) \cong {}_V\mathrm{Inf}_{V/K}\mathrm{Ind}_{U/K}(F)\ .$$

We deduce that Y is exprojective. This is a contradiction, as required. □

Proposition 3.5. *The $\mathbb{Z}$-linear map $\mathrm{lin}_G : \mathcal{T}(G) \to T(G)$ is surjective. However, the $\mathbb{Z}[1/p]$-linear map $\mathrm{can}_G : \mathbb{Z}[1/p]T(G) \to \mathbb{Z}[1/p]\mathcal{T}(G)$ need not restrict to a $\mathbb{Z}$-linear map $T(G) \to \mathcal{T}(G)$. Indeed, putting $p = 3$ and $G = \mathrm{SL}_2(3)$, then the isomorphically unique indecomposable non-simple non-projective p-permutation $\mathbb{F}G$-module Y satisfies $3[Q_8, (\mathrm{can}_G[Y])_{Q_8}] = 2[Q_8, X]$, where X is the isomorphically unique 2-dimensional simple $\mathbb{F}Q_8$-module.*

Proof. Since every 1-dimensional $\mathbb{F}G$-module is exprojective, the surjectivity of the $\mathbb{Z}$-linear map lin_G follows from Boltje [3, 4.7]. Routine techniques confirm the counter-example. □

4 The $\mathbb{K}$-semisimplicity of the commutative algebra $\mathbb{K}\mathcal{T}(G)$

Let $\mathcal{I}(G)$ be the G-set of pairs (P, s) where P is a p-subgroup of G and s is a p'-element of $N_G(P)/P$. Choosing and fixing an arbitrary isomorphism between a suitable torsion subgroup of $\mathbb{K} - \{0\}$ and a suitable torsion subgroup of $\mathbb{F} - \{0\}$, we can understand Brauer characters of $\mathbb{F}G$-modules to have values in $\mathbb{K}$. For a p'-element $s \in G$, we define a species $\epsilon_{1,s}^G$ of $\mathbb{K}T(G)$, we mean, an algebra map $\mathbb{K}T(G) \to \mathbb{K}$, such that $\epsilon_{1,s}^G[M]$ is the value, at s, of the Brauer character of a p-permutation $\mathbb{F}G$-module M. Generally, for $(P, s) \in \mathcal{I}(G)$, we define a species $\epsilon_{P,s}^G$ of $\mathbb{K}T(G)$ such that $\epsilon_{P,s}^G[M] = \epsilon_{1,s}^{N_G(P)/P}[M(P)]$, where $M(P)$ denotes the P-relative Brauer quotient of M^P. The next result, well-known, can be found in Bouc–Thévenaz [5, 2.18, 2.19].

Theorem 4.1. *Given $(P, s), (P', s') \in \mathcal{I}(G)$, then $\epsilon_{P,s}^G = \epsilon_{P',s'}^G$ if and only if we have G-conjugacy $(P, s) =_G (P', s')$. The set $\{\epsilon_{P,s}^G : (P, s) \in_G \mathcal{I}(G)\}$ is the set of species of $\mathbb{K}T(G)$ and it is also a basis for the dual space of $\mathbb{K}T(G)$. The dual basis $\{e_{P,s}^G : (P, s) \in_G \mathcal{I}(G)\}$ is the set of primitive idempotents of $\mathbb{K}T(G)$. As a direct sum of trivial algebras over $\mathbb{K}$, we have*

$$\mathbb{K}T(G) = \bigoplus_{(P,s) \in_G \mathcal{I}(G)} \mathbb{K}e_{P,s}^G\ .$$

Let $\mathcal{J}(G)$ be the G-set of pairs (L,t) where L is a p'-residue-free normal subgroup of G and t is a p'-element of G/L. We define a species $\epsilon_G^{L,t}$ of $\mathbb{K}T^{\rm ex}(G)$ such that, given an indecomposable exprojective $\mathbb{F}G$-module M, then $\epsilon_G^{L,t}[M] = 0$ unless M is the inflation of an $\mathbb{F}G/L$-module $\overline{M}$, in which case, $\epsilon_G^{L,t}$ is the value, at t, of the Brauer character of $\overline{M}$. It is easy to show that, given a p-subgroup $P \leq G$ and a p'-element $s \in N_G(P)/P$, then $\epsilon_{P,s}^G[M] = \epsilon_G^{L,t}[M]$ for all exprojective $\mathbb{F}G$-modules M if and only if L is the normal closure of P in G and t is conjugate to the image of s in G/L. Hence, via the latest theorem, we obtain the following lemma.

Lemma 4.2. *Given $(L,t),(L',t') \in \mathcal{J}(G)$, then $\epsilon_G^{L,t} = \epsilon_G^{L',t'}$ if and only if $L = L'$ and $t =_{G/L} t'$, in other words, $(L,t) =_G (L',t')$. The set $\{\epsilon_G^{L,t} : (L,t) \in_G \mathcal{J}(G)\}$ is the set of species of $\mathbb{K}T^{\rm ex}(G)$ and it is also a basis for the dual space of $\mathbb{K}T^{\rm ex}(G)$.*

Let $\mathcal{K}(G)$ be the G-set of triples (V,L,t) where $V \leq G$ and $(L,t) \in \mathcal{J}(V)$. Given $(L,t) \in \mathcal{J}(G)$, we define a species $\epsilon_{G,L,t}^G$ of $\mathbb{K}\mathcal{T}(G)$ such that, for x as in Section 3,

$$\epsilon_{G,L,t}^G(x) = \epsilon_G^{L,t}(x_G)\ .$$

Generally, for $(V,L,t) \in \mathcal{K}(G)$, we define a species $\epsilon_{V,L,t}^G$ of $\mathbb{K}\mathcal{T}(G)$ such that

$$\epsilon_{V,L,t}^G(x) = \epsilon_{V,L,t}^V({}_V\mathrm{res}_G(x))\ .$$

Using Lemma 4.2, a straightforward adaptation of the argument in [5, 2.18] gives the next result.

Theorem 4.3. *Given $(V,L,t),(V',L',t') \in \mathcal{K}(G)$, then $\epsilon_{V,L,t}^G = \epsilon_{V',L',t'}^G$ if and only if $(V,L,t) =_G (V',L',t')$. The set $\{\epsilon_{V,L,t}^G : (V,L,t) \in_G \mathcal{K}(G)\}$ is the set of species of $\mathbb{K}\mathcal{T}(G)$ and it is also a basis for the dual space of $\mathbb{K}\mathcal{T}(G)$. The dual basis $\{e_{V,L,t}^G : (V,L,t) \in_G \mathcal{K}(G)\}$ is the set of primitive idempotents of $\mathbb{K}\mathcal{T}(G)$. As a direct sum of trivial algebras over $\mathbb{K}$, we have*

$$\mathbb{K}\mathcal{T}(G) = \bigoplus_{(V,L,t)\in_G\mathcal{K}(G)} \mathbb{K}e_{V,L,t}^G\ .$$

We have the following easy corollary on lifts of the primitive idempotents $e_{P,s}^G$.

Corollary 4.4. *Given $(P,s) \in \mathcal{I}(G)$, then $e_{\langle P,s\rangle,P,s}^G$ is the unique primitive idempotent e of $\mathbb{K}\mathcal{T}(G)$ such that $\mathrm{lin}_G(e) = e_{P,s}^G$.*

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