

Archive for
MATH 527, Topics in representation theory, Spring 2025

Bilkent University, Laurence Barker, 11 June 2025.

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Course specification

MATH 527, *Topics in Representation Theory*, Spring 2025

Laurence Barker, Bilkent University. Version: 30 January 2025

Course Aims: To supply some general purpose classical theory of rings and modules, then to see how that theory adapts to modern category theoretic scenarios.

Course Description: We shall begin with some standard ring theory: semisimple rings, the Jacobson radical, algebras over fields or over principal ideal domains. We shall then explore some functorial adaptations of the material, including the theory of Mackey functors, theory of biset functors, categories of spans and cospans, categories of setwise correspondences.

Course Requirements: Basic knowledge of algebra, equivalent to MATH 523 Algebra 1. (We shall be touching on some finite group representation theory as examples of group functors, but we shall cover what we need and we shall avoid delving into the technicalities of character theory and block theory.)

Instructor: Laurence Barker, Office SA 129, barker at fen dot bilkent dot edu dot tr.

Texts:

Required textbook:

- T. Y. Lam, “A First Course in Noncommutative Rings”, Springer, 1991.

Recommended texts:

- Serge Bouc, “Biset functors for Finite Groups”, (Springer, 2010).

See also supplementary material on my homepage

<http://www.fen.bilkent.edu.tr/~barker/fenis2.html>

and on Moodle.

Some journal articles:

A classic paper on functorial methods in group representation theory:

- Jacques Thévenaz, Peter Webb, *The structure of Mackey functors*, Transactions of the American Mathematical Society **347**, 1865-1961 (1995).

A paper applying similar methods to the combinatorial notion of a correspondence:

- Serge Bouc, Jacques Thévenaz, *Correspondence functors and finiteness conditions*, Journal of Algebra **495**, 150-198 (2018).

A curious paper on some essentially similar functorial methods in computer science:

- Brendan Fong, Fabio Zanasi, *Universal constructions for (co)relations: categories, monoidal categories and props*, Logical Methods in computer science **14**(3:14), pp 1-25 (2018).

Classes: Mondays 15:30 - 17:20, Thursdays 10:30 - 11:20, room SA-Z02.

Office Hours: Thursdays, 11:30 - 12:20, in my office, room SA-129.

Weekly Syllabus

The format below is, *Week number; Monday date; Subtopics.*

- 1:** 27 Jan: Ring theory
- 2:** 3 Feb: Ring theory.
- 3:** 10 Feb: Ring theory.
- 4:** 17 Feb: Mackey functors.
- 5:** 24 Feb: Mackey functors.
- 6:** 3 Mar: Biset functors.
- 7:** 10 Mar, *only Thursday*: Biset functors.
- 8:** 17 Mar: Simple and projective functors.
- 9:** 24 Mar: Simple and projective functors.
- 10:** 31 Mar, *only Thursday*: Spans, cospans, pushouts, pullbacks.
- 11:** 7 Apr: Spans, cospans, pushouts, pullbacks.
- 12:** 14 Apr: Correspondence functors.
- 13:** 21 Apr: Correspondence functors.
- 14:** 28 Apr, *only Monday*: Presentations.
- 15:** 5 May: Presentations.
- 16:** 12 May, *only Tuesday*: Review.

Assessment:

A minimum of 75% attendance is required.

- Homework, 10%.
- Midterm, 35%.
- Presentations, 20%.
- Final, 35%.

Presentations: These are to consist of a 25-minute oral talk and a several page written report. The topic is to be broad, algebraic in nature. The pitch of both the talk and the report is to be accessible and fully comprehensible to the audience.

Presentations

Mahmut Esat Akın, *A general theory of canonical induction formulae.*

Mert Akman, *p -Permutation modules and Alperin's Conjecture.*

Metin Can Aydemir, *Representations of $\mathrm{SL}_2(\mathbb{R})$, part 2.*

Muhammet Boran, *Representations of $\mathrm{SL}_2(\mathbb{R})$, part 1.*

Doğan Serkan, *Hopf algebras in representation theory.*

Zeyd Bahadır Kırçu, *Biset functors.*

Mehmet Kırtışoğlu, *Contractibility of the orbit space of the p -subgroup complex.*

Homework and Solutions

MATH 527, *Topics in Representation Theory*, Spring 2025, Laurence Barker

version: 10 May 2025

Homework 1:

Exercise 1.1: Let Δ be a division ring. Show that the dimension of a Δ -module, as a cardinal number, is well-defined. (Hint: Use Zorn's Lemma.)

Homework 2:

Exercise 2.1: Prove the following lemma. It is a standard result that plays a vital role in the theory of semisimple rings. Given a unital ring A , then $A^\circ \cong \text{End}_A({}_A A)$, where A° denotes the opposite algebra of A .

Exercise 2.2: Let F be any field. Give an example of a finite-dimensional unital algebra A over F , an idempotent e of A and a simple eAe -module V such that the induced A -module $\text{ind}_e(V)$ is not simple.

Homework 3:

Exercise 3.1: Show that $\mathbb{C}D_8 \cong \mathbb{C}Q_8$ and $\mathbb{R}D_8 \not\cong \mathbb{R}D_8$.

Exercise 3.2: Up to isomorphism, how many 5-dimensional semisimple algebras are there

(a) Over $\mathbb{C}$?

(a) Over $\mathbb{R}$?

(a) Over $\mathbb{Q}$? (Warning: this last one could be viewed as a trick question.)

Solutions to some homeworks

Some of the solutions are to be presented only in lectures.

1.1: We shall presume two items from the theory of cardinal numbers, as follows:

Proposition A: Given an infinite set S and, for each $s \in S$, a countable set T_s , then $|\bigcup_{s \in S} T_s| \leq |S|$. One way to prove this is first to show that every infinite set can be expressed as a disjoint union of countably infinite sets.

Proposition B: Trichotomy for cardinal numbers: given sets S and T , then $|S| < |T|$ or $|S| = |T|$ or $|S| \geq |T|$. A standard proof of this is based on the theorem asserting that every set can be well-ordered. We mention that this requires the Axiom of Choice.

We shall show that, given a Δ -module M , a Δ -independent subset I of M and a Δ -spanning subset J of M , then $|I| \leq |J|$. We may assume that J is infinite, since the case of finite J is standard undergraduate linear algebra. By Proposition B, we may assume, for a contradiction, that $|I| > |J|$. By an easy Zorn's Lemma argument, we can extend I to a Δ -basis for M ,

so we may assume that I is a Δ -basis for M . For each $j \in J$, let I_j be a finite subset of I such that j is a Δ -linear combination of elements of I_j . Let $\mathcal{I} = \bigcup_j I_j$. By Proposition A, $|\mathcal{I}| = |J|$. Our hypothesis on $|I|$ and $|J|$ implies that there exists some $i \in I - \mathcal{I}$. But i is a Δ -linear combination of finitely many elements of J , and each of those elements of J is a Δ -linear combination of finitely many elements of $\mathcal{I}$. Therefore i is a Δ -linear combination of finitely many elements of $\mathcal{I}$. This contradicts the linear independence of I . $\square$

2.2: We put

$$A = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} : a, c, d \in F \right\}, \quad e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $eAe \cong F$ and eAe has an isomorphically unique simple module S , which is 1-dimensional. Meanwhile, up to isomorphism, A has exactly 2 simple modules, namely S_1 and S_2 upon which A acts as multiplication by a and d , respectively. Letting M be the A -module consisting of the 2-dimensional column vectors over F , we have a short exact sequence

$$0 \leftarrow S_1 \leftarrow M \leftarrow S_2 \leftarrow 0.$$

Noting that $Ae \cong M$, we see that we have an isomorphism $\text{ind}_e(S) = Ae \otimes_{eAe} S \cong M$ given by

$$\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \otimes s \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

where s is a fixed nonzero element of S and $x, y \in F$.

MATH 527: Topics in Representation Theory



Midterm

28 March 2025, LJB

The duration of the exam is 120 minutes.

Please write your name clearly on every sheet of your script.

The use of red pens or faint pencils is prohibited.

1: (10 marks.) Up to isomorphism, how many 6-dimensional semisimple algebras over $\mathbb{R}$ are there?

2: (30 marks.) Let S_3 denote the symmetric group with order 6. Up to isomorphism, how many simple Mackey functors for S_3 over $\mathbb{C}$ are there? You may assume, without proof, that for any finite group G , the number of simple $\mathbb{C}G$ -modules is equal to the number of conjugacy classes of G .

3: (30 marks.) **(a)** Without using any theory of the Jacobson radical, show that any simple finite-dimensional algebra over a field is semisimple.

(b) Give an example of a simple ring that is not semisimple.

4: (30 marks.) Let R be a locally unital ring. Show that every R -module is semisimple if and only if, for every idempotent e of R , the unital ring eRe is semisimple.

Solutions to Midterm

1: Any finite-dimensional semisimple algebra over $\mathbb{R}$ has the form of a finite direct sum

$$\bigoplus_i \text{Mat}_{a_i}(\mathbb{R}) \oplus \bigoplus_j \text{Mat}_{b_j}(\mathbb{C}) \oplus \bigoplus_k \text{Mat}_{c_k}(\mathbb{H})$$

which has dimension

$$\sum_i a_i^2 + 2 \sum_j b_j^2 + 4 \sum_k c_k^2.$$

Taking the dimension to be 6, the possibilities for $(a_1, \dots; b_1, \dots; c_1, \dots)$ are, up to rearrangement,

$$(0; 1; 1), (1, 1; 0; 1), (; 3;), (1, 1; 2;), (2; 1;), (1, 1, 1, 1, 1, 1; 1;), (2, 1, 1; ;), (1, 1, 1, 1, 1, 1; ;) .$$

So the answer is 8.

2: For any finite group G , the simple Mackey functors for G over $\mathbb{C}$ have the form $S_{H,V}$, with no repetitions, where H runs over representatives of the conjugacy classes of subgroups of G and V runs over representatives of the isomorphism classes of $\mathbb{C}N_G(H)/H$ -modules. For each H , the number of possible V is $k(N_G(H)/H)$, where k indicates the number of conjugacy classes. Up to conjugacy, the indices H are $1, C_2, C_3, S_3$. Respectively, the values of $N_G(H)/H$ are $S_3, 1, C_2, 1$ and the numbers of conjugacy classes are 3, 1, 2, 1. So the answer is $3 + 1 + 2 + 1 = 7$.

3: Part (a). Let A be a simple finite-dimensional algebra over a field. Since A is Artinian, A has a minimal ideal S . As an A -module, S is simple. For any $a \in A$, we have $Sa = 0$ or $Sa \cong S$ as A -modules. Let $I = \sum_{a \in A} Sa$. Since I is an ideal of A and A is simple, we have $I = A$. So ${}_A A$ is a sum of simple A -modules. That is one criterion for semisimplicity.

Part (b). Let $\mathbb{k}$ be any field and let V be a $\mathbb{k}$ -module of countably infinite dimension. Let $E = \text{End}_{\mathbb{k}}(V)$ and let I be the ideal of E consisting of those endomorphisms that have finite rank. Put $A = E/I$. To show that A is simple, it is enough to show that, given any infinite rank element $\phi \in E$, then there exist $\theta, \psi \in E$ such that $\theta\phi\psi = \text{id}_V$. Let $\{v_n : n \in \mathbb{N}\}$ be a basis for V and let $\{u_n : n \in \mathbb{N}\}$ be a subset of V such that the elements $u_n + \ker(\phi)$ comprise a basis for $V/\ker(\phi)$. Define ψ such that $\psi(v_n) = u_n$. Choose θ such that $\theta(\phi(u_n)) = v_n$. $\square$

4: Suppose every R -module is semisimple. We are to show that, given e and an eRe -module L , then L is semisimple. Define $M = R \otimes_{eRe} L$ as a R -module. Let $\mathcal{S}$ be a set of simple R -submodules of M such that $M = \bigoplus_{S \in \mathcal{S}} S$ summed over $S \in \mathcal{S}$. Then $L \cong \bigoplus_{S \in \mathcal{S}} eS$ and, by the above lemma, each summand eS is a simple eRe -module. So L is semisimple.

Now assume (a). We are to show that any R -module M is semisimple. By a Zorn's Lemma argument, there exists a set $\mathcal{S}$ of simple R -submodules of M such that the sum N of the elements of $\mathcal{S}$ is direct and $\mathcal{S}$ is maximal subject to that condition. For a contradiction, suppose $N < M$. Let f be an idempotent of R such that $fM \not\leq N$. Since fRf is semisimple, there exists a simple fRf -submodule T of fM such that $T \not\leq fN$. Let $S = RT$ as a R -module.

We claim that S is simple. Consider an idempotent e of R . By the above lemma, the claim will follow when we have shown that eS is simple. Let d be an idempotent of R such that $f, e \in dRd$. We have $dRT \cong dRd \otimes_{fRf} T$ as simple dRd -modules. So $eS \cong edRT$ as simple eRe -modules. The claim is established.

Since $fS \geq T \not\leq fN$, we have $S \not\leq N$. But S is simple, so we can form the direct sum $S \oplus N \leq M$. This contradicts the maximality of $\mathcal{S}$. $\square$

MATH 527: Topics in Representation Theory



Final

20 May 2025, LJB

The duration of the exam is 120 minutes.

Please write your name clearly on every sheet of your script.

The use of red pens or faint pencils is prohibited.

1: (25 marks.) For a finite group G , specify and prove a bijective correspondence between the idempotents of the Burnside ring $B(G)$ and the units of $B(G)$.

2: (25 marks.) For finite groups $I \leq H \leq G$, let e_I^H denote the primitive idempotent of $\mathbb{Q}B(H)$ corresponding to the ring homomorphism $\mathbb{Q}B(H) \rightarrow \mathbb{Q}$ given by $[\Omega] \mapsto |\Omega^I|$ for an H -set Ω .

(a) Give a formula for the restriction $\text{res}_H^G(e_I^G)$.

(b) Give a formula for the induction $\text{ind}_H^G(e_I^H)$.

3: (25 marks.) Let $\mathbb{C}\mathbb{C}$ denote the biset category with coefficients in $\mathbb{C}$. Let $\mathcal{X} = \{C_1, C_2\}$, where C_1 and C_2 denote the cyclic groups with orders 1 and 2, respectively. Let $\mathbb{C}\mathbb{C}_{\mathcal{X}}$ denote the full subcategory of $\mathbb{C}\mathbb{C}$ with set of objects $\mathcal{X}$.

(a) Evaluate the dimension $\dim_{\mathbb{C}}(\oplus \mathbb{C}\mathbb{C}_{\mathcal{X}})$.

(b) Classify the simple functors of $\mathbb{C}\mathbb{C}_{\mathcal{X}}$, we mean, the simple modules of $\oplus \mathbb{C}\mathbb{C}_{\mathcal{X}}$. What are their dimensions?

4: (25 marks.) A category $\mathcal{K}$ is called an **EI-category** provided every endomorphism in $\mathcal{K}$ is an isomorphism. For such $\mathcal{K}$ and a field F , we define the **category algebra** $F\mathcal{K}$ to be the algebra over F with a basis identified with the morphisms in $\mathcal{K}$. The composition is by linear extension of the composition in $\mathcal{K}$, the product of two uncomposable morphisms understood to be 0. Suppose $\mathcal{K}$ has only finitely many morphisms. State and prove a classification of the simple functors of $F\mathcal{K}$, up to isomorphism.

Sketch solutions to Final

1: For each $H \leq G$, let e_H^G denote the primitive idempotent of $\mathbb{Q}B(G)$ associated with the species given by counting the H -fixed points of a G -set. The $\mathbb{Z}$ -span $\beta(G)$ of the e_H^G is a ring and, as such, $\beta(G)$ is isomorphic to a direct sum of copies of the field $\mathbb{F}_2$. Plainly, the idempotents e of $\beta(G)$ and the units u of $\beta(G)$ are in a bijective correspondence whereby $e \leftrightarrow u$ provided $1 - 2e = u$. Since $B(G)$ is a subring of $\beta(G)$, the same formula characterizes a bijective correspondence between the idempotents e of $B(G)$ and the units u of $B(G)$.

2: Part (a). Let ϵ_I^H denote the ring homomorphism specified in the question. Given a G -set Γ , then $\epsilon_I^H(I \text{res}_H[\Gamma]) = \epsilon_I^G[\Omega]$. Therefore,

$$\text{res}_H^G(e_I^G) = \sum_J e_J^G$$

where J runs over representatives of the H -conjugacy classes of subgroups of H that are G -conjugate to I .

Part (b). We first evaluate ${}_H\text{ind}_G(e_H^H)$. For $K \leq G$, we have

$${}_K\text{res}_G\text{ind}_H(e_H^H) = \sum_{KgH \subseteq G} {}_K\text{ind}_{K \cap {}^gH} \text{con}_{K^g \cap H}^g \text{res}_H(e_H^H) .$$

We only get contributions when $K^g \geq H$. So

$${}_K\text{res}_G\text{ind}_H(e_H^H) = \sum_{KgH \subseteq G : K^g \geq H} {}_K\text{ind}_{{}^gH}(e_{{}^gH}^{{}^gH})$$

which vanishes unless $K =_G H$. So

$${}_G\text{ind}_H(e_H^H) = |N_G(H) : H| e_H^G .$$

Turning now to the general case, we have $e_I^H = {}_H\text{ind}_I(e_I^I)/|N_H(I) : I|$, hence

$${}_G\text{ind}_H(e_I^H) = \frac{|N_G(I)|}{|N_H(I)|} e_I^G .$$

3: Part (a). Recall that, for finite groups F and G , the $\mathbb{C}$ -module $\mathbb{C}\mathcal{C}(F, G)$ has a basis consisting of the isomorphism classes of transitive F - G -bisets. So the dimension $d(F, G) = \dim_{\mathbb{C}}(\mathbb{C}\mathcal{C}(F, G))$ is the number of isomorphism classes of subgroups of $F \times G$. In particular,

$$d(C_1, C_1) = 1 , \quad d(C_1, C_2) = d(C_2, C_1) = 2 , \quad d(C_2, C_2) = 5 .$$

Therefore, $\dim_{\mathbb{C}}({}^{\oplus}\mathbb{C}\mathcal{C}\mathcal{X}) = 1 + 2 + 2 + 5 = 10$.

Part (b). We have $\text{Out}(C_1) \cong \text{Out}(C_2) \cong C_1$. So there are exactly 2 isomorphism classes of simple ${}^{\oplus}\mathbb{C}\mathcal{C}\mathcal{X}$ -modules, namely $S_{C_1, \mathbb{C}}$ and $S_{C_2, \mathbb{C}}$, where the subscript $\mathbb{C}$ indicates a trivial module. Plainly, $\dim_{\mathbb{C}}(S_{C_2, \mathbb{C}}) = 1$. To complete the answer, we shall show that $\dim_{\mathbb{C}}(S_{C_1, \mathbb{C}}) = 3$. It is not hard to show that ${}^{\oplus}\mathbb{C}\mathcal{C}\mathcal{X}$ has a simple module S such that, given $G \in \mathcal{X}$, then $S(G)$ is the $\mathbb{C}$ -module of functions $G \rightarrow \mathbb{C}$. Since $S(C_1) \cong S_{C_1, \mathbb{C}}(C_1)$, we have $S \cong S_{C_1, \mathbb{C}}$. Since $\dim_{\mathbb{C}}(S(C_1)) = 1$ and $\dim_{\mathbb{C}}(S(C_2)) = 2$, we have $\dim_{\mathbb{C}}(S) = 1 + 2 = 3$.

Comment: Since $1^2 + 3^2 = 10$, it follows that ${}^{\oplus}\mathbb{C}\mathcal{C}\mathcal{X}$ is semisimple.

4: We can partially order the isomorphism classes $[X]$ of $\mathcal{K}$ -objects X such that $[X'] \leq [X]$ provided there exists a morphism $X' \leftarrow X$. Consider the pairs (X, V) , where X is a $\mathcal{K}$ -object and V is a simple $\mathbb{F}\text{Aut}_{\mathcal{K}}(X)$ -module. Two such pairs (X', V') and (X, V) are to be deemed equivalent provided there is an isomorphism $X' \leftarrow X$ transporting to V' from V . We shall describe a bijective correspondence $[S] \leftarrow [X, V]$ between the isomorphism classes $[S]$ of simple $F\mathcal{K}$ -functors S and the equivalence classes $[X, V]$ of pairs (X, V) . The correspondence, as we shall show, is characterized by the condition that $S(X) \cong V$ and X is minimal subject to the condition $S(X) \neq 0$.

Abusing notation, we let $F\mathcal{K}$ denote the algebra over F such that $F\mathcal{K}$ has a basis consisting of the $\mathcal{K}$ -morphisms. We identify the $F\mathcal{K}$ -functors with the $F\mathcal{K}$ -modules. By considering a Morita equivalence, we reduce to the case where $\mathcal{K}$ has a unique object in each isomorphism class. The Jacobson radical $J(F\mathcal{K})$ is the span of those $\mathcal{K}$ -morphisms that are not endomorphisms. We have

$$F\mathcal{K}/J(F\mathcal{K}) \cong \bigoplus_X F\text{Out}_{\mathcal{K}}(X)$$

summed over the $\mathcal{K}$ -objects X . The described correspondence is now clear.