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MATH 527, Topics in Representation Theory

Bilkent University, Spring 2016, Laurence Barker

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MATH 527, Topics in Representation Theory, Spring 2016

Course specification

Laurence Barker, Mathematics Department, Bilkent University.

version: 20 May 2016.

Course Aims: To supply students with algebraic techniques associated with representations of algebras, especially group algebras of finite groups.

Course Description: This is a second course on finite group representation theory. The focus is on core techniques of modular representation theory.

Course Requirements: MATH 525 *Group Representations* or equivalent.

Instructor: Laurence Barker, Office SAZ 129,
e-mail: barker at fen dot bilkent dot edu dot tr.

Course Texts:
• Required, Peter Webb, "A Course in Finite Group Representation Theory", (preprint 2016).
• Useful for an early part of the course, Michael J. Collins, *Representations and Characters of Finite Groups*, (Cambridge University Press, 1990).

Classes: Tuesdays 10:40 - 12:30 room B-107, Fridays 08:40 - 09:30 room B-107.

Office Hours: Fridays 09:40 - 10:30, room B-107 or Office SA-129.

MATH 527, Spring 2016, **Syllabus**

Week number: Monday date: subtopics.

1: 25 Jan: Induction, restriction, inflation, deflation.

2: 1 Feb: Review of construction of ordinary character tables.

3: 8 Feb: Proofs of Burnside $p^\alpha q^\beta$-Theorem and Frobenius' Theorem on Frobenius groups.

4: 15 Feb: Mackey Decomposition Theorem, Clifford's Theorem.

5: 22 Feb: Jacobson radical. Modular representations of finite p-groups.

6: 29 Feb: Projective modules and primitive idempotents for finite-dimensional algebras.

7: 7 Mar: Projective modules of group algebras.

8: 14 Mar: Splitting fields. The decomposition map. The cde triangle.

9: 21 Mar: Blocks of defect zero.

10: 28 Mar: Brauer characters, decomposition matrix, orthonormality relations.

11: 4 Apr: Calculating Brauer characters.

12: 11 Apr: The Krull–Schmidt Theorem. Vertices, sources and the Green correspondence.

13: 18 Apr: Defect groups of G-Algebras. Defect groups of a block of a group algebra. The Brauer morphism. The Brauer Correspondence.

14: 25 Apr: Defect theory for interior G-algebras. Source algebras.

15: 2 May: Review.

MATH 527, Spring 2016, **Assessment**

• Homework 15%, Midterm 25%, Presentations, 25%, Final, 35%.

• 75% attendance is compulsory.

Presentations:

Merve Acar, *Ordinary and modular characters of the non-abelian group of order pq.*

Elif Doğan Dar, *Frobenius and symmetric algebras.*

Muhammed Said Gündoğan, *The Green correspondence.*

Çisil Karagüzel, *On some of the simple composition factors of the p-permutation ring as a biset functor.*

Hatice Mutlu, *Brauer pairs.*

İsmail Alpereh Öğüt, *Classifying the simple functors of the bifree monomial Burnside category.*

Mustafa Anıl Tokmak, *Real representations and the Frobenius–Schur indicator.*

Melih Üçer, *Blocks and defect groups.*

Homework Exercises:

1: Find the ordinary character tables of the groups A_6 and S_6.

2: Let $H \leq G$ be finite groups, let ψ be a $\mathbb{C}H$-character and let χ be a $\mathbb{C}G$-character. The following equality is called **Frobenius Reciprocity**. Prove the equality.

$$\langle \psi \,|\, {}_H\mathrm{res}_G(\chi) \rangle_H = \langle {}_G\mathrm{ind}_H(\psi) \,|\, \chi \rangle_G \ .$$

3: For a finite group G and an irreducible $\mathbb{C}G$-character χ, show that $\chi(1)$ divides $|G : Z(G)|$.

4: Let $\mathbb{L}$ be a field an G a finite group. Show that $\mathbb{L}G$ is semisimple if and only if the characteristic of $\mathbb{L}$ does not divide the order of G.

5: Let $H \leq G \geq K$ be finite groups, R a commutative unital ring, U an RH-module, V and RK-module. Find a formula for the RG-module ${}_G\mathrm{Ind}_H(U) \otimes_R {}_G\mathrm{Ind}_K(V)$.

6: Find a counter-example to the statement, "Given $K \trianglelefteq G$, an algebraically closed field $\mathbb{F}$ of prime characteristic p and an indecomposable $\mathbb{F}G$-module M, then ${}_K\mathrm{Res}_G(M)$ is a direct sum of mutually conjugate $\mathbb{F}K$-modules".

Homeworks from Textbook:

Ordinary character theory: Chapter 3, Exercises 3, 4, 5, 6.

Modular Representations: Chapter 5, Exercises 4, 9. Chapter 6, Exercises 8, 10.

Modular characters and decomposition numbers: Chapter 8, Exercise 4. Chapter 9, Exercises 5, 11. Chapter 10, Exercise 3.

MATH 527: *Topics in Representation Theory.* **Practice Final**

LJB, 6 May 2016, Bilkent.

Please put your name on every sheet of your manuscript.
Marks may be deducted for excessive length of solutions.

1: 25% The simple group $\mathrm{PSL}_3(2) \cong \mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$ of order 168 has the following ordinary character table, where $\mu = (-1 + i\sqrt{7})/2 = \zeta + \zeta^2 + \zeta^4$ and $\nu = (-1 - i\sqrt{7})/2 = \zeta^3 + \zeta^5 + \zeta^6$ with $\zeta = e^{2\pi i/7}$. Find the modular character table, decomposition matrix and Cartan matrix in characteristic 2. (Hint: consider the actions of Galois automorphisms the 7-th roots of unity.)

	$G = \mathrm{PSL}_3(2)$	1	21	56	42	24	24	$\lvert[g]\rvert$
	$\chi(g)$	1	2	3	4	7	7	$\lvert\langle g\rangle\rvert$
χ	χ_0	1	1	1	1	1	1	
	χ_1	3	-1	0	1	μ	ν	
	χ_2	3	-1	0	1	ν	μ	
	χ_3	6	2	0	0	-1	-1	
	χ_4	7	-1	1	-1	0	0	
	χ_5	8	0	-1	0	1	1	

2: 25% Let $\mathbb{F}$ be a field of prime characteristic p. Let K be a normal subgroup of a finite group G. Show that the number of isomorphism classes of simple $\mathbb{F}G$-modules is greater than or equal to the number of G-conjugacy classes of isomorphism classes of simple $\mathbb{F}K$-modules.

3: 25% Let $\mathbb{F}$, p, K, G be as in Question 2. Now assume that the index $|G : K|$ is a power of p. Let M be a 1-dimensional $\mathbb{F}K$-module. Without using Green's Indecomposability Criterion, give a quick proof that the induced $\mathbb{F}G$-module ${}_G\mathrm{Ind}_K(M)$ is indecomposable.

4: 25% Let A be a finite-dimensional algebra over a field $\mathbb{L}$. A **symmetrizing form** on A is defined to be a linear map $\sigma : A \to \mathbb{L}$ such that $\ker(\sigma)$ contains no two-sided ideal and $\sigma(ab) = \sigma(ba)$ for all $a, b \in A$.

(a) Show that, given a symmetrizing form on A, then $\ker(\sigma)$ contains no left ideal and no right ideal of A.

(b) Show that, given an idempotent e of A, then any symmetrizing form on A restricts to a symmetrizing form on the algebra eAe.

Solutions to MATH 527, Practice Final, Spring 2016.

1: Let $\mathbb{F}$ be a sufficiently large field of characteristic 2. Noting that the number of 2-regular conjugacy classes is 4, also noting that the degree of χ_5 is equal to and perforce divisible by the 2-part of $|G|$, we see that the irreducible $\mathbb{F}G$-characters can be numbered as ϕ_0, ϕ_1, ϕ_2, ϕ_3 where χ_0 and χ_5 are the restrictions of ϕ_0 and ϕ_3. Since G is simple, $\phi_0(1) = 1$ and $\phi_1(1) \geq 2 \leq \phi_2(1)$. Without loss of generality, χ_1 restricts to ϕ_1 or $\phi_0 + \phi_1$. The latter case is impossible, because it implies that ϕ_1 has at least 3 distinct Galois conjugates. Therefore χ_1 and χ_2 restrict to ϕ_1 and ϕ_2. We have established the following 2-modular character table.

$G = \mathrm{PSL}_3(2)$		1	21	56	42	24	24	$\lvert[g]\rvert$
	$\chi(g)$	1	2	3	4	7	7	$\lvert\langle g\rangle\rvert$
ϕ	ϕ_0	1	–	1	–	1	1	
	ϕ_1	3	–	0	–	μ	ν	
	ϕ_2	3	–	0	–	ν	μ	
	ϕ_3	8	–	-1	–	1	1	

Immediately, we obtain the following decomposition matrix D and Cartan matrix C

D	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5
ϕ_0	1	0	0	0	1	0
ϕ_1	0	1	0	1	1	0
ϕ_2	0	0	1	1	1	0
ϕ_3	0	0	0	0	0	1

C	ϕ_0	ϕ_1	ϕ_2	ϕ_3
ϕ_0	2	1	1	0
ϕ_1	1	3	2	0
ϕ_2	1	2	3	0
ϕ_2	0	0	0	1

2: Sketch: Apply Clifford's Theorem.

3: Sketch: Restrict to Sylow p-subgroups and apply Frobenius Reciprocity.

4: Part (a). For a contradiction, let I be a non-zero left ideal of A such that $I \subseteq \ker(\sigma)$. Choose a non-zero element $b \in I$. Since $AaA \not\subseteq \ker(\sigma)$. there exist $a, c \in A$ such that $abc \notin \ker(\sigma)$. Then $\sigma(cab) = \sigma(abc) \neq 0$. But $cab \in I$, contradicting the assumption on I.

Part (b). For a contradiction, let J be a non-zero ideal of eAe annihilated by σ. Since σ does not annihilate the ideal $AeJeA$ of A, there exist $a, b \in A$ and $j \in J$ such that $0 \neq \sigma(aejeb)$. Then $0 \neq \sigma(ejeba) = \sigma(e^2jeba) = \sigma(ejebae) = \sigma(eje.ebae)$, contradicting the hypothesis on J.

MATH 527: *Topics in Representation Theory.* Midterm

LJB, 22 March 2016, Bilkent.

Please put your name on every sheet of your manuscript.

Warning: For each question, the length of the solution must be equivalent to, at most, one page in handwriting of the size of this text, with plenty of whitespace. Beyond that length, all excess writing will be ignored.

1: 20% Let K be a field and let C_9 denote the cyclic group of order 9. Up to isomorphism, how many simple KG-modules are there:

(a) when $K = \mathbb{C}$?

(b) when $K = \mathbb{R}$?

(c) when $K = \mathbb{Q}$?

2: 20% Let G be a finite group. Suppose there exists a faithful irreducible $\mathbb{C}G$-character χ.

(a) Show that the centre $Z(G)$ is cyclic.

(b) Let $H \leq G$ and suppose that ${}_H\mathrm{res}_G(\chi)$ is irreducible. Show that $C_G(H) = Z(G)$.

3: 20% The semidihedral group of order 16 has presentation

$$\mathrm{SD}_{16} = \langle a, b : a^8 = b^2 = 1, bab^{-1} = a^3 \rangle \ .$$

Find the ordinary character table of SD_{16}, clearly explaining your method.

4: 20% Let E be the elementary abelian group of order 9. Let $G = Q_8 \ltimes E$ as a semidirect product, where Q_8 acts on E with kernel $Z(Q_8)$. Find the degrees of the irreducible $\mathbb{C}G$-characters and the multiplicity of each degree. (Warning: Avoid routines that would be too long for the examiner to read.)

5: 20% (This apparently arbitrary problem arises as an important lemma in the study of p-permutation functors.) Let p be a prime and m an integer coprime to p. Let R be the cyclic group with order m, let P be an elementary abelian p-group and let $G = R \ltimes P$ where the action of R on P is such that, writing $\mathbb{F}_p$ for the field of order p, regarding P as an $\mathbb{F}R$-module, then $P \cong \bigoplus_S n_S S$ where S runs over representatives of the isomorphism classes of the simple $\mathbb{F}_p R$-modules and $n_S = 1$ unless S is trivial, in which case $n_S \in \{0, 2\}$. Note that every automorphism α of G fixes P and hence induces an automorphism $\overline{\alpha}$ of R. Show that the group homomorphism $\mathrm{Aut}(G) \ni \alpha \mapsto \overline{\alpha} \in \mathrm{Aut}(R)$ is surjective. (Hint: writing $|P| = p^r$, embed R in $\mathrm{GL}_r(p)$ and consider the normalizer of R in $\mathrm{GL}_r(p)$. Also compare P with the regular $\mathbb{F}_p R$-module.)

MATH 527: *Topics in Representation Theory.* **Final**

LJB, 11 May 2016, Bilkent.

Attention: This exam paper has two pages, not just one page!

Please put your name on every sheet of your manuscript.
Marks may be deducted for excessive length of solutions.

1: 25% The Mathieu group M_{10}, of order 720, has a normal subgroup isomorphic to A_6. The ordinary character table of M_{10} is as shown. The first 5 conjugacy classes are those conjugacy classes that are contained in the copy of A_6.

	$G = M_{10}$	1	45	80	90	144	180	90	90	$\|[g]\|$
	$\chi(g)$	1	2	3	4	5	4	8	8	$\|\langle g\rangle\|$
χ	χ_0	1	1	1	1	1	1	1	1	
	χ_1	1	1	1	1	1	-1	-1	-1	
	χ_2	9	1	0	1	-1	1	-1	-1	
	χ_3	9	1	0	1	-1	-1	1	1	
	χ_4	10	2	1	-2	0	0	0	0	
	χ_5	10	-2	1	0	0	0	$i\sqrt{2}$	$-i\sqrt{2}$	
	χ_6	10	-2	1	0	0	0	$-i\sqrt{2}$	$i\sqrt{2}$	
	χ_7	16	0	-2	0	1	0	0	0	

Find the 2-modular irreducible characters, the decomposition matrix and the Cartan matrix. You may assume that A_6 has irreducible 2-modular characters ψ and ψ' as specified in the next table.

$G = A_6$	1	45	40	40	90	72	72	$\|[g]\|$
$\chi(g)$	1	2	3	3	4	5	5	$\|\langle g\rangle\|$
ψ	4	$-$	1	-2	$-$	-1	-1	
ψ'	4	$-$	-2	1	$-$	-1	-1	

2: 25% Let $\mathbb{F}$ be an algebraically close field of prime characteristic p, let G be a finite group with a normal subgroup K such that $|K|$ is not divisible by p. Let S and T be simple $\mathbb{F}G$-modules such that there exists a non-split short exact sequence of $\mathbb{F}G$-modules $1 \to S \to M \to T \to 0$. Show that, if K acts trivially on S, then K acts trivially on T.

3: 25% Let $\pi : R \to S$ be a surjective homomorphism of commutative unital rings. Let $H \leq G \geq K$ be finite groups. We write RG/H for the permutation RG-module that has a basis identifiable with the set of left cosets $\{gH : g \in G\}$.

(a) For $x \in G$, we define the **suborbit map** associated with x to be the R-linear map $\theta_x^R : RG/H \to RG/K$ sending gH to the sum of those cosets fK that are contained in $gHxK$. Show that the suborbit maps θ_x^R comprise an R-basis for the R-module $\mathrm{Hom}_{RG}(RG/H, RG/K)$.

(b) Let Ω and Γ be G-sets. Show that regarding S as an R-module via π, reduction from coefficients in R to coefficients in S yields a surjective R-map $\mathrm{Hom}_{RG}(R\Omega, R\Gamma) \to \mathrm{Hom}_{SG}(S\Omega, S\Gamma)$.

(c) Show that reduction to coefficients in S induces a bijective correspondence from isomorphism classes of indecomposable direct summands of permutation RG-modules to isomorphism classes of indecomposable direct summands of permutation SG-modules.

(d) Give an example of R, S, G and an R-module M such that reduction to S does not yield a surjective R-map $\mathrm{End}_R(M) \to \mathrm{End}_S(S \otimes_R M)$.

4: 25% Let $\mathbb{E}$ be a finite-degree extension of a field $\mathbb{F}$. Let A be an algebra over $\mathbb{F}$ (not necessarily finite-dimensional) and let U and V be A-modules (not necessarily finite-dimensional). Define $\mathbb{E}A = \mathbb{E} \otimes_{\mathbb{F}} A$ and define $\mathbb{E}U$ and $\mathbb{E}V$ similarly. Show that there is a unique $\mathbb{E}$-linear isomorphism $\mathbb{E}\mathrm{Hom}_A(U, V) \to \mathrm{Hom}_{\mathbb{E}A}(\mathbb{E}U, \mathbb{E}V)$ such that $1 \otimes \theta \mapsto (1 \otimes u \mapsto 1 \otimes \theta(u))$ for $\theta \in \mathrm{Hom}_A(U, V)$ and $u \in U$. (Warning: In view of the example in part (d) of Question 3, some care is needed.)

MATH 527, Solutions to Final, Spring 2016.

1: The 2-modular character table is as shown.

$G = M_{10}$	1	45	80	90	144	180	90	90	$\lvert[g]\rvert$
$\chi(g)$	1	2	3	4	5	4	8	8	$\lvert\langle g\rangle\rvert$
ϕ ϕ_0	1	–	1	–	1	–	–	–	
ϕ_1	8	–	-1	–	-2	–	–	–	
ϕ_2	16	–	-1	–	-1	–	–	–	

Indeed, since the number of 2-regular conjugacy classes of M_{10} is exactly 3, there are exactly 3 irreducible 2-modular characters. Plainly, ϕ_1 is such a character. By considering the 2-parts of $|M_{10}$ and $\chi_7(1)$, we see that ϕ_2 is such a character. Let ϕ be the remaining irreducible 2-modular character, and let ϕ_1 be the class function as specified in the table. Since ϕ_2 must restrict to a sum of 1 or 2 irreducible 2-modular characters of A_6, neither of the characters ψ and ψ' can appear in the restriction of ϕ_2. Nor can ψ or ψ' appear in the restriction of ϕ_0. Therefore, both ψ and ψ' must appear in the restriction of ϕ. On the other hand, by Frobenius reciprocity, ϕ must appear in the induced characters of ψ and ψ'. We deduce that ${}_{A_6}\mathrm{res}_{M_{10}}(\phi) = \psi + \psi'$. This equality determines ϕ because all the conjugacy classes in $M_{10} - A_6$ are 2-singular. We deduce that $\phi = \phi_1$.

It is now easy to see that the decomposition and Cartan matrices are as shown.

D	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
ϕ_0	1	1	1	1	2	2	2	0
ϕ_1	0	0	1	1	1	1	1	0
ϕ_2	0	0	0	0	0	0	0	1

C	ϕ_0	ϕ_1	ϕ_2
ϕ_0	16	8	0
ϕ_1	8	5	0
ϕ_2	0	0	1

2: Sketch: The central idempotent of $\mathbb{F}K$ associated with the trivial $\mathbb{F}K$-module is also a central idempotent of $\mathbb{F}G$.

3: Sketch: Parts (a), (b), (c) are straightforward. For part (d), let p be an odd prime, let $R = \mathbb{Z}$, let $S = \mathbb{Z}/p$ and let $G = C_2$. Let $M = M_1 \oplus M_2$ where M_1 and M_2 are the two isomorphically distinct 1-dimensional RG-modules. There cannot exist a surjection from $\mathrm{End}_R(M)$ to $\mathrm{End}_S(S \otimes M)$ because the former endomorphism algebra has R-rank 2 whereas the latter has S-rank 4.

Comment: Notwithstanding the observation that $\mathbb{Q} \otimes_{\mathbb{Z}} M$ is the regular $\mathbb{Q}C_2$-module, parts (b) and (d) are not in conflict. The $\mathbb{Z}G$-module M is not a permutation module.

4: Sketch: Let $\{\epsilon_i : i \in I\}$ be an $\mathbb{F}$-basis for $\mathbb{E}$ and write $\theta(1 \otimes u) = \sum_i \epsilon_i \otimes \theta_i(u)$. It is easy to see that each $\theta_i \in \mathrm{Hom}_A(U, V)$.