Course Aims: To supply students with algebraic techniques associated with representations of algebras, especially group algebras of finite groups.

Course Description: This is a second course on finite group representation theory. The focus is on core techniques of modular representation theory.

Course Requirements: MATH 525 Group Representations or equivalent.

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Course Texts:
• Useful for an early part of the course, Michael J. Collins, Representations and Characters of Finite Groups, (Cambridge University Press, 1990).

Classes: Tuesdays 10:40 - 12:30 room B-107, Fridays 08:40 - 09:30 room B-107.

Office Hours: Fridays 09:40 - 10:30, room B-107 or Office SA-129.
MATH 527, Spring 2016, Syllabus

Week number: Monday date: subtopics.

1: 25 Jan: Induction, restriction, inflation, deflation.

2: 1 Feb: Review of construction of ordinary character tables.

3: 8 Feb: Proofs of Burnside $p^\alpha q^\beta$-Theorem and Frobenius' Theorem on Frobenius groups.

4: 15 Feb: Mackey Decomposition Theorem, Clifford's Theorem.

5: 22 Feb: Jacobson radical. Modular representations of finite $p$-groups.

6: 29 Feb: Projective modules and primitive idempotents for finite-dimensional algebras.

7: 7 Mar: Projective modules of group algebras.

8: 14 Mar: Splitting fields. The decomposition map. The cde triangle.


10: 28 Mar: Brauer characters, decomposition matrix, orthonormality relations.

11: 4 Apr: Calculating Brauer characters.


15: 2 May: Review.
Homeworks from Textbook:

Ordinary character theory: Chapter 3, Exercises 3, 4, 5, 6.

Modular Representations: Chapter 5, Exercises 4, 9. Chapter 6, Exercises 8, 10.

1: 25% The simple group $\text{PSL}_3(2) \cong \text{GL}_3(2) \cong \text{PSL}_2(7)$ of order 168 has the following ordinary character table, where $\mu = \frac{-1 + i\sqrt{7}}{2} = \zeta + \zeta^2 + \zeta^4$ and $\nu = \frac{-1 - i\sqrt{7}}{2} = \zeta^3 + \zeta^5 + \zeta^6$ with $\zeta = e^{2\pi i/7}$. Find the modular character table, decomposition matrix and Cartan matrix in characteristic 2. (Hint: consider the actions of Galois automorphisms the 7-th roots of unity.)

| $G = \text{PSL}_3(2)$ | 1  | 21  | 56  | 42  | 24  | 24  | $\langle |g| \rangle$ |
|-----------------------|----|-----|-----|-----|-----|-----|-------------------|
| $\chi$                | $\chi_0$ | 1  | 1   | 1   | 1   | 1   | 1  |
|                       | $\chi_1$ | 3  | -1  | 0   | 1   | $\mu$ | $\nu$ |
|                       | $\chi_2$ | 3  | -1  | 0   | 1   | $\nu$ | $\mu$ |
|                       | $\chi_3$ | 6  | 2   | 0   | 0   | -1  | -1  |
|                       | $\chi_4$ | 7  | -1  | 1   | -1  | 0   | 0   |
|                       | $\chi_5$ | 8  | 0   | -1  | 0   | 1   | 1   |

2: 25% Let $\mathbb{F}$ be a field of prime characteristic $p$. Let $K$ be a normal subgroup of a finite group $G$. Show that the number of isomorphism classes of simple $\mathbb{F}G$-modules is greater than or equal to the number of $G$-conjugacy classes of isomorphism classes of simple $\mathbb{F}K$-modules.

3: 25% Let $\mathbb{F}$, $p$, $K$, $G$ be as in Question 2. Now assume that the index $|G : K|$ is a power of $p$. Let $M$ be a 1-dimensional $\mathbb{F}K$-module. Without using Green’s Indecomposability Criterion, give a quick proof that the induced $\mathbb{F}G$-module $\text{Ind}_K^G(M)$ is indecomposable.

4: 25% Let $A$ be a finite-dimensional algebra over a field $L$. A symmetrizing form on $A$ is defined to be a linear map $\sigma : A \rightarrow L$ such that $\ker(\sigma)$ contains no two-sided ideal and $\sigma(ab) = \sigma(ba)$ for all $a, b \in A$.

(a) Show that, given a symmetrizing form on $A$, then $\ker(\sigma)$ contains no left ideal and no right ideal of $A$.

(b) Show that, given an idempotent $e$ of $A$, then any symmetrizing form on $A$ restricts to a symmetrizing form on the algebra $eAe$. 

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Please put your name on every sheet of your manuscript. Marks may be deducted for excessive length of solutions.
1: Let $\mathbb{F}$ be a sufficiently large field of characteristic 2. Noting that the number of 2-regular conjugacy classes is 4, also noting that the degree of $\chi_5$ is equal to and perforce divisible by the 2-part of $|G|$, we see that the irreducible $\mathbb{F}G$-characters can be numbered as $\phi_0$, $\phi_1$, $\phi_2$, $\phi_3$ where $\chi_0$ and $\chi_5$ are the restrictions of $\phi_0$ and $\phi_3$. Since $G$ is simple, $\phi_0(1) = 1$ and $\phi_1(1) \geq 2 \leq \phi_2(1)$. Without loss of generality, $\chi_1$ restricts to $\phi_1$ or $\phi_0 + \phi_1$. The latter case is impossible, because it implies that $\phi_1$ has at least 3 distinct Galois conjugates. Therefore $\chi_1$ and $\chi_2$ restrict to $\phi_1$ and $\phi_2$. We have established the following 2-modular character table.

| $G = \text{PSL}_3(2)$ | $\chi(g)$ | 1 | 2 | 3 | 4 | 7 | 7 | $||g||$ | $||\langle g \rangle||$ |
|------------------------|------------|---|---|---|---|---|---|----------|-----------------|
| $\phi$                 | $\phi_0$   | 1 | - | 1 | - | 1 | 1 | $||g||$   | $||\langle g \rangle||$ |
| $\phi_1$               | $\phi_1$   | 3 | - | 0 | - | $\mu$ | $\nu$ | $||g||$   | $||\langle g \rangle||$ |
| $\phi_2$               | $\phi_2$   | 3 | - | 0 | - | $\nu$ | $\mu$ | $||g||$   | $||\langle g \rangle||$ |
| $\phi_3$               | $\phi_3$   | 8 | - | -1 | - | 1 | 1 | $||g||$   | $||\langle g \rangle||$ |

Immediately, we obtain the following decomposition matrix $D$ and Cartan matrix $C$

$$
D = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad
C = \begin{bmatrix}
\phi_0 & \phi_1 & \phi_2 & \phi_3 \\
2 & 1 & 1 & 0 \\
1 & 3 & 2 & 0 \\
1 & 2 & 3 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

2: Sketch: Apply Clifford’s Theorem.

3: Sketch: Restrict to Sylow $p$-subgroups and apply Frobenius Reciprocity.

4: Part (a). For a contradiction, let $I$ be a non-zero left ideal of $A$ such that $I \subseteq \ker(\sigma)$. Choose a non-zero element $b \in I$. Since $AaA \not\subseteq \ker(\sigma)$, there exist $a, c \in A$ such that $abc \not\in \ker(\sigma)$. Then $\sigma(cab) = \sigma(abc) \neq 0$. But $cab \in I$, contradicting the assumption on $I$.

Part (b). For a contradiction, let $J$ be a non-zero ideal of $eAe$ annihilated by $\sigma$. Since $\sigma$ does not annihilate the ideal $AeJeA$ of $A$, there exist $a, b \in A$ and $j \in J$ such that $0 \neq \sigma(ejeba)$. Then $0 \neq \sigma(ejeba) = \sigma(e^2jeba) = \sigma(ejebae) = \sigma(eje.ebae)$, contradicting the hypothesis on $J$. 

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MATH 527: Topics in Representation Theory. Midterm

LJB, 22 March 2016, Bilkent.

Please put your name on every sheet of your manuscript.

Warning: For each question, the length of the solution must be equivalent to, at most, one page in handwriting of the size of this text, with plenty of whitespace. Beyond that length, all excess writing will be ignored.

1: 20% Let $K$ be a field and let $C_9$ denote the cyclic group of order 9. Up to isomorphism, how many simple $K G$-modules are there:

(a) when $K = \mathbb{C}$?
(b) when $K = \mathbb{R}$?
(c) when $K = \mathbb{Q}$?

2: 20% Let $G$ be a finite group. Suppose there exists a faithful irreducible $\mathbb{C} G$-character $\chi$. 

(a) Show that the centre $Z(G)$ is cyclic.
(b) Let $H \leq G$ and suppose that $H \text{res}_G(\chi)$ is irreducible. Show that $C_G(H) = Z(G)$.

3: 20% The semidihedral group of order 16 has presentation

$$SD_{16} = \langle a, b : a^8 = b^2 = 1, bab^{-1} = a^3 \rangle.$$ 

Find the ordinary character table of $SD_{16}$, clearly explaining your method.

4: 20% Let $E$ be the elementary abelian group of order 9. Let $G = Q_8 \rtimes E$ as a semidirect product, where $Q_8$ acts on $E$ with kernel $Z(Q_8)$. Find the degrees of the irreducible $\mathbb{C} G$-characters and the multiplicity of each degree. (Warning: Avoid routines that would be too long for the examiner to read.)

5: 20% (This apparently arbitrary problem arises as an important lemma in the study of $p$-permutation functors.) Let $p$ be a prime and $m$ an integer coprime to $p$. Let $R$ be the cyclic group with order $m$, let $P$ be an elementary abelian $p$-group and let $G = R \rtimes P$ where the action of $R$ on $P$ is such that, writing $\mathbb{F}_p$ for the field of order $p$, regarding $P$ as an $\mathbb{F} R$-module, then $P \cong \bigoplus S n_S S$ where $S$ runs over representatives of the isomorphism classes of the simple $\mathbb{F}_p R$-modules and $n_S = 1$ unless $S$ is trivial, in which case $n_S \in \{0, 2\}$. Note that every automorphism $\alpha$ of $G$ fixes $P$ and hence induces an automorphism $\overline{\alpha}$ of $R$. Show that the group homomorphism $\text{Aut}(G) \ni \alpha \mapsto \overline{\alpha} \in \text{Aut}(R)$ is surjective. (Hint: writing $|P| = p^r$, embed $R$ in $\text{GL}_r(p)$ and consider the normalizer of $R$ in $\text{GL}_r(p)$. Also compare $P$ with the regular $\mathbb{F}_p R$-module.)
Attention: This exam paper has two pages, not just one page!

Please put your name on every sheet of your manuscript.
Marks may be deducted for excessive length of solutions.

1: 25% The Mathieu group $M_{10}$, of order 720, has a normal subgroup isomorphic to $A_6$. The ordinary character table of $M_{10}$ is as shown. The first 5 conjugacy classes are those conjugacy classes that are contained in the copy of $A_6$.

\[
\begin{array}{cccccccc}
G = M_{10} & 1 & 45 & 80 & 90 & 144 & 180 & 90 & 90 \\
\chi(g) & 1 & 2 & 3 & 4 & 5 & 4 & 8 & 8 \\
\chi & \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \\
\end{array}
\]

Find the 2-modular irreducible characters, the decomposition matrix and the Cartan matrix. You may assume that $A_6$ has irreducible 2-modular characters $\psi$ and $\psi'$ as specified in the next table.

\[
\begin{array}{cccccccc}
G = A_6 & 1 & 45 & 40 & 40 & 90 & 72 & 72 \\
\chi(g) & 1 & 2 & 3 & 3 & 4 & 5 & 5 \\
\psi & 4 & - & 1 & -2 & - & -1 & -1 \\
\psi' & 4 & - & -2 & 1 & - & -1 & -1 \\
\end{array}
\]

2: 25% Let $F$ be an algebraically close field of prime characteristic $p$, let $G$ be a finite group with a normal subgroup $K$ such that $|K|$ is not divisible by $p$. Let $S$ and $T$ be simple $FG$-modules such that there exists a non-split short exact sequence of $FG$-modules $1 \to S \to M \to T \to 0$. Show that, if $K$ acts trivially on $S$, then $K$ acts trivially on $T$.

3: 25% Let $\pi : R \to S$ be a surjective homomorphism of commutative unital rings. Let $H \leq G \geq K$ be finite groups. We write $RG/H$ for the permutation $RG$-module that has a basis identifiable with the set of left cosets $\{gH : g \in G\}$.

(a) For $x \in G$, we define the suborbit map associated with $x$ to be the $R$-linear map $\theta_R^x : RG/H \to RG/K$ sending $gH$ to the sum of those cosets $fK$ that are contained in $gHxK$. Show that the suborbit maps $\theta_R^x$ comprise an $R$-basis for the $R$-module $\text{Hom}_{RG}(RG/H, RG/K)$.

(b) Let $\Omega$ and $\Gamma$ be $G$-sets. Show that regarding $S$ as an $R$-module via $\pi$, reduction from coefficients in $R$ to coefficients in $S$ yields a surjective $R$-map $\text{Hom}_{RG}(R\Omega, R\Gamma) \to \text{Hom}_{SG}(S\Omega, S\Gamma)$.
(c) Show that reduction to coefficients in $S$ induces a bijective correspondence from isomorphism classes of indecomposable direct summands of permutation $RG$-modules to isomorphism classes of indecomposable direct summands of permutation $SG$-modules.

(d) Give an example of $R$, $S$, $G$ and an $R$-module $M$ such that reduction to $S$ does not yield a surjective $R$-map $\text{End}_R(M) \to \text{End}_S(M \otimes_R M)$.

4: 25% Let $E$ be a finite-degree extension of a field $F$. Let $A$ be an algebra over $F$ (not necessarily finite-dimensional) and let $U$ and $V$ be $A$-modules (not necessarily finite-dimensional). Define $E_A = E \otimes_F A$ and define $EU$ and $EV$ similarly. Show that there is a unique $E$-linear isomorphism $E\text{Hom}_A(U,V) \to \text{Hom}_{E_A}(EU, EV)$ such that $1 \otimes \theta \mapsto (1 \otimes u \mapsto 1 \otimes \theta(u))$ for $\theta \in \text{Hom}_A(U,V)$ and $u \in U$. (Warning: In view of the example in part (d) of Question 3, some care is needed.)
1: The 2-modular character table is as shown.

\[
\begin{array}{cccccc|cc}
G = M_{10} & 1 & 45 & 80 & 90 & 144 & 180 & 90 & 90 \\
\chi(g) & 1 & 2 & 3 & 4 & 5 & 4 & 8 & 8 \\
\phi & \phi_0 & 1 & -1 & 1 & -1 & - & - & - \\
\phi_1 & 8 & -1 & -1 & 1 & 1 & -2 & - & - \\
\phi_2 & 16 & -1 & -1 & -1 & - & - & - & - \\
\end{array}
\]

Indeed, since the number of 2-regular conjugacy classes of $M_{10}$ is exactly 3, there are exactly 3 irreducible 2-modular characters. Plainly, $\phi_1$ is such a character. By considering the 2-parts of $|M_{10}$ and $\chi_7(1)$, we see that $\phi_2$ is such a character. Let $\phi$ be the remaining irreducible 2-modular character, and let $\phi_1$ be the class function as specified in the table. Since $\phi_2$ must restrict to a sum of 1 or 2 irreducible 2-modular characters of $A_6$, neither of the characters $\psi$ or $\psi'$ can appear in the restriction of $\phi_2$. Nor can $\psi$ or $\psi'$ appear in the restriction of $\phi_0$. Therefore, both $\psi$ and $\psi'$ must appear in the restriction of $\phi$. On the other hand, by Frobenius reciprocity, $\phi$ must appear in the induced characters of $\psi$ and $\psi'$. We deduce that $A_6 \res_{M_{10}}(\phi) = \psi + \psi'$. This equality determines $\phi$ because all the conjugacy classes in $M_{10} - A_6$ are 2-singular. We deduce that $\phi = \phi_1$.

It is now easy to see that the decomposition and Cartan matrices are as shown.

\[
\begin{array}{ccccccc}
D & \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \\
\phi_0 & 1 & 1 & 1 & 1 & 2 & 2 & 0 & 0 \\
\phi_1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
\phi_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

2: Sketch: The central idempotent of $FK$ associated with the trivial $FK$-module is also a central idempotent of $FG$.

3: Sketch: Parts (a), (b), (c) are straightforward. For part (d), let $p$ be an odd prime, let $R = \mathbb{Z}$, let $S = \mathbb{Z}/p$ and let $G = C_2$. Let $M = M_1 \oplus M_2$ where $M_1$ and $M_2$ are the two isomorphically distinct 1-dimensional $RG$-modules. There cannot exist a surjection from $\text{End}_R(M)$ to $\text{End}_S(S \otimes M)$ because the former endomorphism algebra has $R$-rank 2 whereas the latter has $S$-rank 4.

Comment: Notwithstanding the observation that $\mathbb{Q} \otimes_{\mathbb{Z}} M$ is the regular $\mathbb{Q}C_2$-module, parts (b) and (d) are not in conflict. The $\mathbb{Z}G$-module $M$ is not a permutation module.

4: Sketch: Let $\{\epsilon_i : i \in I\}$ be an $F$-basis for $E$ and write $\theta(1 \otimes u) = \sum_i \epsilon_i \otimes \theta_i(u)$. It is easy to see that each $\theta_i \in \text{Hom}_A(U, V)$. 

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