Archive of past papers, solutions and homeworks for MATH 525, Group representations, Spring 2012, LJB

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MATH 525 Group Representations, Spring 2012

Handout 1: Course specification

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version: 23 February 2012.

Course aims: To acquire knowledge and skill in group representation theory and related areas of algebra.

Course instructor: Laurence Barker, Office SAZ 129.

Course description: The course will consist of three segments.

• Some abstract ring theory. This material is of general value in itself and, in particular, it supplies a foundation for group representation theory. The main results are the Artin–Wedderburn Theorem on semisimple rings, the various characterizations of the Jacobson radical and the resulting description of the structure of an artinian ring, the Hopkin's Levitsky Theorem asserting that every artinian ring is noetherian.

• Character theory of finite groups, or equivalently, representation theory of finite groups over a field of characteristic zero. After studying the theory and techniques behind the construction of character tables, we shall give two classic applications: proofs of Burnside's $p^\alpha q^\beta$-Theorem and Frobenius' Theorem concerning subgroups $H \leq G$ such that $H \cap {}^x H = 1$ for all $x \in G - H$.

• Presentations and free discussion on modular representation theory. The emphasis of this segment will depend on the interests of the class and on the amount of time available.

The victory condition for this course is skill at techniques for constructing character tables and a good grasp of the theory behind those techniques, including the underlying ring theory.

Course texts:

J. L. Alperin, R. B. Bell, "Groups and Representations", (Springer, Berlin, 1995); for quick but incomplete treatment of the ring theory and introductory account of the character theory.

M. J. Collins, "Representations and Characters of Finite Groups", Cambridge Studies in Adv. Math. **22**, (Cambridge, Univ. Press, 1990); more advanced character theory, mainly for specialists, but with lucid proofs of Burnside's $p^\alpha q^\beta$-Theorem and Frobenius' Theorem.

C. W. Curtis, I. Reiner, "Methods of Representation Theory with Applications to Finite Groups and Orders", (Wiley, New York, 1981), classic compendious reference for representation theory.

G. James, M. Liebeck, "Representations and characters of groups", (Cambridge Univ. Press, 1993); for more detailed account of the character theory, with many examples.

T. Y. Lam, "A First Course in Noncommutative rings", Graduate Texts in Math. 131, (Springer, Berlin, 1991); for thorough treatment of the ring theory and some preliminaries on ordinary character theory and modular theory.

G. Navarro, "Characters and Blocks of Finite Groups", LMS Lecture Notes **250**, (Cambridge Univ. Press, 1998); for the modular representation theory.

Classes: Wednesdays 13:40 - 15:30 SAZ 02, Fridays 15:40 - 16:30, SAZ 02.

Office Hours: Fridays, 16:40 - 17:30, SAZ 129.

Assessment:

- Homeworks, Quizzes, Presentations, 20%.
- Midterm I, 25%, FISH date.
- Midterm II, 25%, FISH date.
- Final, 30%.

Syllabus: Week number: Monday date, subtopics.

1: 6 Feb. Group representations. Group algebras.

2: 13 Feb. Semisimple rings and Artin–Wedderburn Theorem.

3: 20 Feb. Algebras. Artinian rings. Noetherian rings.

4: 27 Feb. The Jacobson radical and the Hopkins–Levitsky Theorem.

5: 5 Mar. Ordinary characters. The character table. Some easy examples.

6: 12 Mar. Further easy examples. Midterm 1, 14th March, Wednesday, 13:40.

7: 19 Mar. Centrally primitive idempotents of the ordinary group algebra. Orthogonality properties of the character table.

8: 26 Mar. Induction, restriction and inflation of characters.

9: 2 Apr. Further techniques for constructing character tables.

10: 9 Apr. Integrality conditions on the character table. Burnside's $p^\alpha q^\beta$-Theorem and Frobenius' Theorem.

11: 16 Apr. Modular characters. Midterm 2, 18th April, Wednesday, 13:40.

12: 23 Apr. Blocks and central characters. Orthogonality relations for modular characters. Decomposition and Cartan matrices.

13: 30 Apr. Vertices of indecomposable modules. Defect groups of blocks. Brauer's First Main Theorem.

14: May. Presentations.

15: May. Review.

Homeworks and presentations

Homework 1

1: Adapting the Three Isomorphism Theorems for groups, state three analogous theorems for modules. Assuming the group-theoretic versions of those theorems, prove all three module-theoretic versions simultaneously, in at most two sentences.

2: Let F be a field, and let $\mathrm{Mat}_{\mathbb{N}}(F)$ be the ring consisting of the matrices whose rows and columns are indexed by $\mathbb{N}$, with only finitely many non-zero entries in each column. Let I be the ideal of $\mathrm{Mat}_{\mathbb{N}}(F)$ consisting of those matrices which have only finitely many non-zero entries. Show that the ring $\mathrm{Mat}_{\mathbb{N}}(F)/I$ is simple but not semisimple.

Solutions to Homework 1:

1: (Last part only.) By undergraduate group theory, the three specified isomorphisms exist, as isomorphisms of abelian groups. It is easy to check that each of the three isomorphisms commutes with the actions of the ring.

2: The solution, here, is based on Lam, pages 43, 44. Let $\Lambda = \mathrm{Mat}_{\mathbb{N}}(F)$. We shall show that, given an element $g \in \Lambda - I$, then the ideal of Λ generated by g owns the unity element of Λ. It will then follow that I is a maximal ideal of Λ, in other words, Λ/I is simple.

Let V be the Λ-module consisting of the column vectors over F with coordinates indexed by $\mathbb{N}$. Let $\{e_n : n \in \mathbb{N}\}$ be the standard basis for V. We mean to say, the n-th coordinate of e_n is 1 and all the other coordinates are 0. The representation $\Lambda \to \mathrm{End}_F(V)$ is plainly a ring isomorphism.

Since infinitely many of the entries of g are non-zero, the subspace $g(V)$ is of infinite-dimensional. But V is of countable dimension, hence $g(V)$ is of countable dimension and we can choose a basis $\{d_n : n \in \mathbb{N}\}$ of $g(V)$. For each n, let $c_n \in V$ such that $g(c_n) = d_n$. Let h be the endomorphism of V such that $h(e_n) = c_n$. Let f be the map $g(V) \to V$ such that $f(d_n) = e_n$. By Remark 1.3, we can choose a subspace U of V such that $V = g(V) \oplus U$ We can extend f to an endomorphism of V by letting f annihilate U. Via the isomorphism in the previous paragraph, we can regard f and h as matrices. Then $fgh(e_n) = fg(c_n) = f(d_n) = e_n$. In other words, $fgh = 1$. We have shown that Λ/I is simple.

To show that Λ/I is not semisimple, consider a basis $\{w_{n,m} : n, m \in \mathbb{N}\}$ for V. Let I_j be the left ideal of Λ consisting of those matrices f such that $f(w_{n,m}) = 0$ when $n \geq j$. We have $I_0 + I \leq I_1 + I \leq ...$ as a chain of left ideals in Λ/I, in other words, as a chain of submodules of the regular module ${}_{\Lambda/I}\Lambda/I$. We clam that each $I_j + I < I_{j+1} + I$. Supposing that $I_j + I = I_{j+1} + I$ for some j, let f be the element of I_{j+1} such that $f(w_{n,m}) = w_{n,m}$ when $n \leq j$. Writing $f = g + h$ with $g \in I_j$ and $h \in I$, then each $g(w_{j,m}) = 0$ hence each $h(w_{j,m}) = 1$. But that is impossible because only finitely many of the entries of h are non-zero. The claim is established.

On the other hand, if Λ/I is semisimple, then ${}_{\Lambda/I}\Lambda/I$ is a direct sum of finitely many simple modules and ${}_{\Lambda/I}\Lambda/I$ cannot have an infinite strictly ascending chain of submodules. Therefore Λ/I is not semisimple.

Homework 2

1: Consider the symmetric group S_4, the alternating group A_5 and the non-abelian group F_{21} with order 21. For each of these groups, find as much as you can of the ordinary character tables using *only* the following techniques:
• classification of the simple characters of a finite abelian group,
• inflation from quotient groups,
• realization of the given group as a symmetry group of a polytope, for instance, a cube, a dodecahedron, a simplex.

Homework 3

1: Using the equivalence of two characterizations of a hermitian matrix, show that the row othonormality property of character tables is equivalent to the column orthonormality property.

Homework 4

1: Complete the proof of Frobenius Recioprocity (following class discussion).

2: Show that the symmetric square and alternating square of a character are given by the specified formulas (following class discussion).

3: Find the character tables of A_5 and S_5 (especially making use of induction from subgroups as discussed in class).

Presentations

Cihan Bahran, *The Jacobson radical.*

Merve Demirel, *The modular characters of* A_5.

Emrah Karagöz, *The representations of the symmetric group, Part 2.*

Serkan Sakar, *The epresentations of* SL_2.

Emre Şen, *The representations of the symmetric group, Part 1.*

Yasemin Büyükçülük, *The structure of the character ring.*

V. Dağhan Yaylıoğlu, *Defect groups of blocks.*

MATH 525: Group Representations. **Midterm 1**. LJB, 16 March 2012, Bilkent.

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript.

1: 20% Without proof, give three equivalent definitions of a **semisimple module** of a unital ring and give three equivalent definitions of a **semisimple ring** in terms of its left modules. Without proof, state the Artin–Wedderburn Structure Theorem for semisimple rings. Briefly explain why the definitions of a semisimple ring would still refer to the same class of rings if they were to be expressed in terms of right modules instead of left modules.

2: 20% Let p be a prime. Let $\mathbb{F}_p$ denote the field with order p. Find the number of isomorphism classes of 8-dimensional non-commutative semisimple algebras over $\mathbb{F}_p$. (Recall that, for each positive integer a, there is a field, unique up to isomorphism, with order p^a. Wedderburn's Little Theorem asserts that every finite division ring is a field.)

3: 20% Let A be a unital ring. Let M be a finitely-generated A-module such that every simple A-module occurs as a composition factor of M. Let $E = \mathrm{End}_A(M)$. Show that, if A is semisimple, then $\mathrm{End}_E(M) \cong A$.

4: 20% Let F be a field. Let $\mathfrak{A}$ be the ring consisting of the matrices having the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ where $a, b \in F$. Show that, up to isomorphism, $\mathfrak{A}$ has a unique simple module. Deduce that, in the notation of the previous question, if we drop the assumption that A is semisimple, then the conclusion can fail and the ring $\mathfrak{A}$ is a counter-example.

5: 20% Recall that an **idempotent** of a ring R is an element $i \in R$ such that $i^2 = i \neq 0$. Show that, if R is a semisimple, then the ring $iRi = \{iri : r \in R\}$ is semisimple.

Solutions to Midterm 1

1: Let R be a unital ring. An R-module M is said to be **semisimple** provided it satisfies the following three equivalent conditions: M is a sum of simple modules; M is a direct sum of simple modules, every submodule of M is a direct summand of M. The ring R is said to be **semisimple** provided it satisfies the following three equivalent definitions: every left R-module is semisimple, every finitely generated left R-module is semisimple, the regular R-module ${}_R R$ is semisimple.

The Artin–Wedderburn Theorem asserts that R is semisimple if and only if

$$R \cong \mathrm{Mat}_{n_1}(\Delta_1) \oplus \ldots \oplus \mathrm{Mat}_{n_k}(\Delta_k)$$

as a direct sum of full matrix algebras over division rings Δ_j.

The theorem implies that R is semisimple if and only if the opposite ring R° is semisimple. But left R-modules can be identified with right R°-modules. So the definition of a semisimple ring remains unchanged when expressed in terms of right modules instead of left modules.

2: By the Artin–Wedderburn Theorem, together with the recollections in the question, any finite-dimensional semisimple algebra A over $\mathbb{F}_p$ can be expressed in the form

$$A \cong \mathrm{Mat}_{n_1}(\mathbb{F}_{q_1}) \oplus \ldots \oplus \mathrm{Mat}_{n_k}(\mathbb{F}_{q_k})$$

where each $q_k = p^{a_k}$. By the uniqueness of the Wedderburn decomposition, the pairs of positive integers (n_j, a_j) are unique up to reordering. We have

$$\dim(A) = n_1^2 a_1 + \ldots + n_k^2 a_k \ .$$

Note that A is non-commutative if and only if $n_j \geq 2$ for some index j.

Now putting $\dim(A) = 8$ and supposing that A is non-commutative, the six possibilities are $A = \mathrm{Mat}_2(\mathbb{F}_{p^2})$ and $A = \mathrm{Mat}_2(\mathbb{F}_p) \oplus B$ where $B \cong \mathbb{F}_{q_2} \oplus \ldots \oplus \mathbb{F}_{q_k}$ and

$$(q_2, \ldots, q_k) \in \{(p^4), (p^3, p), (p^2, p^2), (p^2, p, p), (p, p, p, p)\} \ .$$

3: Write $A = A_1 \oplus \ldots \oplus A_k$ where each $A_j \cong \mathrm{Mat}_{n_j}(\Delta_j)$. Let S_j be the isomorphically unique simple A_j-module and let e_j be the unity element of A_j. We have $M = M_1 \oplus \ldots \oplus M_k$ where $M_j = e_j M$ and each $M_j \cong m_j S_j$.

Let $E = \mathrm{End}_A(M)$. Then $E \cong \bigoplus_{i,j} \mathrm{Hom}_A(M_i, M_j)$. Half of Schur's Lemma says that $\mathrm{Hom}_A(S_i, S_j) = 0$ when $i \neq j$. Hence $\mathrm{Hom}_A(M_i, M_j) = 0$ and $E = E_1 \oplus \ldots \oplus E_k$ where $E_j \cong \mathrm{End}_A(M_j)$. The other half of Schur's Lemma says that $\mathrm{End}_A(S_j)$ is a division ring. Furthermore, by standard proofs of the Artin–Wedderburn Theorem, $\Delta_j \cong \mathrm{End}_A(S_j)^\circ$. It follows that $E_j \cong \mathrm{Mat}_{m_j}(\Delta_j^\circ)$. In particular, E is semisimple.

Let $F = \mathrm{End}_E(M)$. Replacing A with E, we deduce that F is semisimple. Each M_j is an E_j-module and $F = F_1 \oplus \ldots \oplus F_k$ where $F_j \cong \mathrm{End}_E(M_j) \cong \mathrm{Mat}_{n'_j}(\Delta_j)$. Finally, $n_j m_j = \dim_{\Delta_j}(M_j) = n'_j m_j$, hence $n_j = m_j$.

Comment: As will be explained in class, this result in Question 3 can be viewed as a version of a theorem called the Double Centralizer Theorem.

4: Any element of $\mathfrak{A}$ can be written uniquely in the form $a1 + b\beta$ where $\beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Note that $\beta^2 = 0$. Given a simple $\mathfrak{A}$-module S, then $\beta S \leq S$. Since S is simple, either $\beta S = 0$ or else $\beta S = S$. But the latter case is impossible because $\beta^2 S = 0$. So $a + b\beta$ acts on S as multiplication by a. So, up to isomorphism, $\mathfrak{A}$ has only one simple module S and, furthermore, $\dim_F(S) = 1$. We have $\mathrm{End}_A(S) \cong F$ which is evidently not isomorphic to A.

5: Write $R = R_1 \oplus ... \oplus R_k$ as a direct sum of sequisimple algebras R_j, and write $i = i_1 + ... + i_k$ with each $i_j \in R_k$. Let S_j be a simple R_j-module and let Δ_j be a division ring such that R_j is isomorphic to a full matrix algebra over Δ_j. Choose a Δ_j-basis for iS_j and extend to a Δ_j-basis for S_j. Imposing coordinates with respect to those bases, we can make an identification $R_j = \mathrm{Mat}_{n_j}(\Delta_j)$. Then i_j is a diagonal matrix whose non-zero entries are all 1. We have $iRi = \bigoplus_j i_j R_j i_j$ summed over those indices j such that $i_j \neq 0$. For each such j, we have $i_j R i_j \cong \mathrm{Mat}_{m_j}(\Delta_j)$ where m_j is the rank of i_j.

MATH 525: Group Representations. Midterm 2. LJB, 20 April 2012, Bilkent.

Time allowed: 110 minutes. Please put your name on EVERY sheet of your manuscript.

1: 10% State (without proof) the row and column orthonormality relations for the ordinary character table of a finite group.

2: 20% Let G be a finite group, let N be a normal subgroup of G with index $|G:N| = 2$ and let ψ be an irreducible $\mathbb{C}N$-character. Show that either $\mathrm{ind}_N^G(\psi)$ is irreducible or else $\mathrm{ind}_N^G(\psi)$ is the sum of 2 distinct irreducible $\mathbb{C}G$-characters.

3: 30% Let G be a finite group, let χ be an irreducible $\mathbb{C}G$-character.

(a) Let $\overline{\chi}$ be the function $G \to \mathbb{C}$ such that $\overline{\chi}(g)$ is the complex conjugate of $\chi(g)$. Briefly, explain why $\overline{\chi}$ is an irreducible $\mathbb{C}G$-character.

(b) Let $\overline{\chi}.\chi$ be the function $G \to \mathbb{C}$ such that $(\overline{\chi}.\chi)(g) = \overline{\chi}(g).\chi(g)$. Briefly, explain why $\overline{\chi}.\chi$ is a $\mathbb{C}G$-character.

(c) Show that $\langle \chi_0 \,|\, \overline{\chi}.\chi \rangle = 1$, where χ_0 denotes the trivial $\mathbb{C}G$-character.

4: 40% The simple group $\mathrm{GL}_3(2)$ has order 168. It has a subgroup isomorphic to S_4. It has a non-abelian subgroup F_{21} with order 21. It has 6 conjugacy classes, with elements of order 1, 2, 3, 4, 7, 7. The sizes of the conjugacy classes are 1, 21, 56, 42, 24, 24, respectively. Construct the ordinary character table of $\mathrm{GL}_3(2)$. (Hint: To obtain some of the irreducible characters, consider the characters induced from the 1-dimensional characters of S_4 and F_{21}. To obtain the remaining two irreducible characters, use orthonormality, avoiding complicated calculations by considering automorphisms of $\mathbb{Q}[e^{2\pi i/7}]$. When checking your calculated values, you may find it convenient to note that $\sum_{a \in \mathbb{Z}/7} e^{2\pi i a^2/7} = i\sqrt{7}$.)

MATH 525: Group Representations. <u>Final</u>. LJB, 18 May 2012, Bilkent.

Time allowed: two hours. Please put your name on EVERY sheet of your manuscript.

1: 25% Let A be a semisimple ring and let i be an idempotent of A. Consider the subring iAi of A. Show that the condition $S \cong iT$ characterizes a bijective correspondence between the isomorphism classes of simple iAi-modules S and the isomorphism classes of those simple A-modules T such that $iT \neq 0$.

2: 25% Consider two semisimple rings

$$A = \bigoplus_{j=1}^{k} \mathrm{Mat}_{n_j}(\Delta_j) \ , \qquad A' = \bigoplus_{j=1}^{k'} \mathrm{Mat}_{n'_j}(\Delta'_j)$$

where each Δ_j and Δ'_j are division rings. Show that if $A \cong A'$ then $k = k'$ and, after renumbering, each $n_j = n'_j$ and each $\Delta_j \cong \Delta'_j$.

3: 25% Assuming Maschke's Theorem and the Artin–Wedderburn Structure Theorem for Semisimple Rings (including the rider to the latter stated in the previous question), show that $\mathbb{C}D_8 \cong \mathbb{C}Q_8$ and $\mathbb{R}D_8 \not\cong \mathbb{R}Q_8$.

4: 25% Let N be a normal subgroup of a finite group G. Let ϕ be an irreducible $\mathbb{C}N$-character. Recall that, for $g \in G$, we write ${}^g\phi$ to denote the irreducible $\mathbb{C}N$-character such that ${}^g\phi({}^gx) = \phi(x)$ for $x \in N$. The subgroup $T = \{g \in G : {}^g\phi = \phi\}$ is called the **inertia group** of ϕ in G. Let ψ be an irreducible $\mathbb{C}T$-character such that ϕ occurs in $\mathrm{res}^T_N(\psi)$.

(a) Show that $\mathrm{res}^T_N(\psi)$ is an integer multiple of ϕ.

(b) Show that $\mathrm{ind}^G_T(\psi)$ is irreducible.

Solutions to Final

1: Let s and s' be two non-zero elements of iT. Since T is simple, there exists an element $a \in A$ such that $s' = as$. But $is' = s'$ and $is = s$, hence $s' = iais$. It follows that iT is simple. By considering the regular modules of iAi and A, we see that every simple iAi-module is has the form iT. Finally, consider simple A-modules T and T' such that $iT \cong iT'$. Writing $1 = e_1 + ... + e_k$ as the sum of the primitive idempotents of $Z(A)$, then $i = ie_1 + ... + ie_k$. So there must exist some j such that ie_j does not annihilate the iAi-module $iT \cong iT'$. Hence $e_jT \neq 0$ and $e_jT' \neq 0$. Since e_j is a primitive idempotent of $Z(A)$, we must have $T \cong T'$.

Alternatively, one can argue by explicitly describing iAi as in Question 4 of Midterm 1.

2: We may identify A and A' via an isomorphism. Write $\bigoplus_j^k A_j = A = A' = \bigoplus_j^{k'} A'_j$ where $A_j \cong \mathrm{Mat}_{n_j}(\Delta_j)$ and $A'_j \cong \mathrm{Mat}_{n'_j}(\Delta'_j)$. Letting e_j be the unity element of A_j, then $A_j = e_jAe_j$. A similar comment holds for the unity element e'_j of A'_j. But the set of primitive idempotents of the centre $Z(A) = Z(A')$ is $\{e_1....,e_k\} = \{e'_1,...,e'_{k'}\}$. Therefore $k = k'$ and, after renumbering, each $e_j = e'_j$ and $A_j = A'_j$. It remains only to show that, given two expressions $\mathrm{Mat}_n(\Delta) \cong R \cong \mathrm{Mat}_{n'}(\Delta')$ for a sesquisimple ring R, where Δ and Δ' are division rings, then $\Delta \cong \Delta'$ and $n = n'$. Letting S be a simple R-module, then the opposite rings of Δ and Δ' are both isomorphic to $\mathrm{End}_R(S)$. Therefore $\Delta \cong \Delta'$. Making an identification $\Delta = \Delta'$ via an isomorphism, we have $n = \dim_\Delta(S) = n'$.

3: By the two specified theorems, together with the algebraic closure of $\mathbb{C}$, both $\mathbb{C}D_8$ and $\mathbb{C}Q_8$ are direct sums of matrix algebras. Both of them have at least 4 mutually non-isomorphic 1-dimensional modules, because $D_8/Z(D_8) \cong V_4 \cong Q_8/Z(Q_8)$. But D_8 and Q_8 are non-abelian. So the 8-dimensional semisimple algebras $\mathbb{C}D_8$ and $\mathbb{C}Q_8$ must both have the form

$$\mathbb{C}D_8 \cong \mathbb{C}^4 \oplus \mathrm{Mat}_2(\mathbb{C}) \cong \mathbb{C}Q_8 \ .$$

Again by inflating from V_4, we see that $\mathbb{R}D_8$ and $\mathbb{R}Q_8$ have at least 4 mutually non-isomorphic 1-dimensional modules. By regarding D_8 as the group of symmetries of a square, we deduce that one of the Wedderburn components of $\mathbb{R}D_8$ is $\mathrm{Mat}_2(\mathbb{R})$. On the other hand, there is an evident ring epimorphism $\mathbb{R}Q_8 \to \mathbb{H}$. Therefore

$$\mathbb{R}D_8 \cong \mathbb{R}^4 \oplus \mathrm{Mat}_2(\mathbb{R}) \ , \qquad \mathbb{R}Q_8 \cong \mathbb{R}^4 \oplus \mathbb{H} \ .$$

Now invoking the conclusion to the previous question, we conclude that $\mathbb{R}D_8 \ncong \mathbb{R}Q_8$.

4: By Frobenius Reciprocity, ψ occurs in $\mathrm{ind}_N^T(\phi)$. So $\mathrm{res}_N^T(\psi)$ is a direct summand of $\mathrm{res}_N^T(\mathrm{ind}_N^T(\phi))$. But

$$(\mathrm{ind}_N^T(\phi))(x) = \sum_{gN \subseteq T} \phi({}^g x) = |T : N| \phi(x)$$

for $x \in N$. Therefore $\mathrm{res}_N^T(\mathrm{ind}_N^T(\phi)) = |T : N|\, \phi$. Part (a) follows. Writing $\mathrm{res}_T^G(\mathrm{ind}_T^G(\psi)) = \psi + \theta$, then $\langle \mathrm{ind}_T^G(\psi)) \,|\, \mathrm{ind}_T^G(\psi)) \rangle = \langle \psi \,|\, \psi + \theta \rangle$. It suffices to show that ψ does not occur in θ. Interchanging g and g^{-1}, we have

$$\theta(x) = \sum_{gN \subseteq G-T} \phi({}^g x) = \sum_{gN \subseteq G-T} {}^g\phi(x)$$

as a sum of conjugates of ϕ which are all distinct from ϕ. Therefore ϕ does not occur in $\mathrm{res}_N^T(\theta)$ and, perforce, ψ does not occur in θ.