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MATH 525, Group Representations, Fall 2099

Bilkent University, Laurence Barker, 14 January 2024

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MATH 525

Group Representations, Fall 2023

Course specification

Laurence Barker, Bilkent University. Version: 20 December 2023

Classes: Mondays 11:30 - 12:20, Wednesdays 15:30 - 17:20, room SA Z02.

Office Hours: Mondays 10:30 - 11:20, SA 129.

Instructor: Laurence Barker
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Course Texts: The primary course text is:

> Peter Webb, "A Course in Finite Group Representation Theory", Cambridge University Press 2016. There is a free PDF download of the prepublication version on the homepage of Peter Webb, University of Minnesota.

For the general ring theory, the recommended text is

> T.-Y. Lam, "A First Course in Noncommutative Rings", (Springer, Berlin, 1991).

For further representation theory, a recommended text is

> Jean-Paul Serre, *Linear Representations of Finite Groups*, (Springer, Berlin, 1977).

Homework: Homeworks will be supplied, sometimes in course notes, sometimes in other files on my homepage. They carry no course credit.

Course Documentation: As the course progresses, further documentation will appear on my homepage.

Syllabus: *Below is a tentative course schedule.* The format of the following details is *Week number: Monday date: Subtopics (Section numbers).*

Syllabus: The format of the following details is *Week number: Monday date: Subtopics.*

1: 11 Sept: Groups, rings, modules, representations.

2: 18 Sept: Groups, rings, modules, representations.

3: 25 Sept: General theory of semisimple rings. Maschke's Theorem.

4: 2 Oct: The group algebra

5: 9 Oct: Irreducible characters of semisimple algebras over a field of characteristic 0, in general.

6: 16 Oct: Ordinary character tables for some small finite groups.

7: 23 Oct: Centrally primitive idempotents of semisimple algebras, and the group algebra in particular.

8: 30 Oct: Orthogonality properties of the ordinary character table.

9: 6 Nov: Functors on modules and characters. Frobenius reciprocity.

10: 13 Nov: Constructing character tables using inflation and induction.

11: 20 Nov: Examples of character tables.

12: 27 Nov: Symmetric and alternating squares. Further groups and their character tables.

13: 4 Dec: Integrality properties of ordinary irreducible characters. Central characters.

14: 11 Dec: Burnside's $p^\alpha q^\beta$-Theorem and characterization of Frobenius groups.

15: 18 Dec: Review.

Assessment:

• Quizzes, 10%,
• Midterm, 45%, at 20:00 - 22:00, Thursday, 16 November, in SA-Z03.
• Final, 45%, at 09:00 on Friday, 22 December, in SA-Z19.

An FZ grade will be awarded for Midterm marks that are below 20%.

75% attendance is compulsory.

MATH 525 Presentations, with visitors, Fall 2023

Venue: Thursday, 21 December 2023, Room SA Z01

09:30: Mert Akman, *Brauer's characterization of ordinary characters.*

10:00: Cabize Kavalcı, *Introduction to modular representation theory.*

10:30: Mehmet Kirtişoğlu, *Independence of projective resolutions for* Ext *and* Tor.

11:00: Esat Akin, *The Stone–von Neumann Theorem for the Heisenberg group associated with a finite abelian group.*

11:30: Sevket Kaan Alkır, *Frobenius algebras.*

— — — [Lunchtime] — — —

13:30: Metehan Akkuş, *Representations of locally compact groups.*

14:00: Muhammed Gökman, *Representations of Lie groups.*

14:30: Enes Koç, *Irreducible representations of* SO(3).

15:00: Onur Ege Erden, *Irreducible representations of* SU(3).

15:30: Cansu Özdemir, *Spin representations of* $2n$*-dimensional rotation groups.*

16:00: Kağan Akman, *Clifford's Theorem.*

16:30: Deniz Özyörük, *Tanaka reconstruction.*

Quizzes, with solutions

MATH 525, *Group Representations*, Fall 2023, Laurence Barker

version: 20 December 2023

Quiz 1: Let $G = C_3 = \{1, a, a^2\}$. Observe that the 1-dimensional $\mathbb{C}$-vector space

$$\mathbb{C} \sum_{g \in G} g = \mathbb{C}(1 + a + a^2)$$

is a $\mathbb{C}G$-submodule of the regular $\mathbb{C}G$module

$${}_{\mathbb{C}G}\mathbb{C}G = \mathbb{C}1 \oplus \mathbb{C}a \oplus \mathbb{C}a^2 \ .$$

Find a basis for a complementary submodule.

Solution: Defining $\omega = e^{2\pi i/3}$, we have

$${}_{\mathbb{C}G}\mathbb{C}G = \mathbb{C}(1 + a + a^2) \oplus \mathbb{C}(1 + \omega^2 a + \omega a^2) \oplus \mathbb{C}(1 + \omega a + \omega^2 a^2)$$

as a direct sum of 1-dimensional $\mathbb{C}G$-modules. So the submodule $\mathbb{C}(1 + a + a^2)$ has complementary submodule $\mathbb{C}(1 + \omega^2 a + \omega a^2) \oplus \mathbb{C}(1 + \omega a + \omega^2 a^2)$. One basis for the complementary submodule is the set $\{1 + \omega^2 a + \omega a^2, 1 + \omega a + \omega^2 a^2\}$.

Another basis for the complementary submodule is $\{1 - 2a + a^2, 1 + a - 2a^2\}$.

Comment 1: The above decomposition of $\mathbb{C}G$ already appeared in the answer to Homework Question 1.1 part (b).

Comment 2: For any finite group G and any field K of characteristic 0, the regular KG-module ${}_{KG}KG$ decomposes as a direct sum of KG-modules

$${}_{KG}KG = K \sum_{g \in G} g \oplus \{ \sum_{g \in G} \lambda_g g : \sum_{g \in G} \lambda_g = 1\} \ .$$

Quiz 2: Up to isomorphism, how many 12-dimensional semisimple algebras over $\mathbb{C}$ are there?

Solution: Since $\mathbb{C}$ is algebraically closed, any semisimple algebra over $\mathbb{C}$ is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$. Therefore, the answer is the number of ways of expressing 12 as a sum of non-increasing squares. The ways of thus expressing 12 are

$$12 = 9 + 3.1 = 3.4 = 2.4 + 4.1 = 4 + 8.1 = 12.1 \ .$$

Therefore, the answer is 5.

Quiz 2: Advanced version: How many 12-dimensional semisimple algebras over $\mathbb{R}$ are there? You may use a theorem of Frobenius which asserts that every finite-dimensional division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.

Solution: Let m denote the answer.

For any natural number n, we define $f(n)$ to be the number of ways of expressing n as a sum of non-increasing squares. A table of values of $f(n)$, for $n \leq 12$, is as follows.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1	1	1	1	2	2	2	2	3	4	4	4	5

Given any division ring Δ, then $f(n)$ is the number of isomorphism classes of n-dimensional algebras over Δ that can be decomposed as direct sums of matrix algebras. Any 12-dimensional algebra A over $\mathbb{R}$ decomposes as $A = A_{\mathbb{H}} \oplus A_{\mathbb{C}} \oplus A_{\mathbb{R}}$ where each A_Δ is a direct sum of matrix algebras over Δ. As parameters of A, we introduce $a = \dim_{\mathbb{H}}(A_{\mathbb{H}})$ and $b = \dim_{\mathbb{C}}(A_{\mathbb{C}})$ and $c = \dim_{\mathbb{R}}(A_{\mathbb{R}})$. We have $4a+2b+c = 12$. For each (a, b, c), the number of possible isomorphism classes for A is $f(a)f(b)f(c)$. Therefore,

$$m = \sum_{a,b,c \in \mathbb{N}\,:\, a+b+c=12} f(a)f(b)f(c) \; .$$

The possibilities for (a, b, c) and the values of $f(a)$, $f(b)$, $f(c)$ and $f(a)f(b)f(c)$ are as shown.

a	3	2	2	2	1	1	1	1	1	0	0	0	0	0	0	0
b	0	2	1	0	4	3	2	1	0	6	5	4	3	2	1	0
c	0	0	2	4	0	2	4	6	8	0	2	4	6	8	10	12
$f(a)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$f(b)$	1	1	1	1	2	1	1	1	1	2	2	2	1	1	1	1
$f(c)$	1	1	1	2	2	1	2	2	3	2	2	4	2	3	4	5
$f(a)f(b)f(c)$	1	1	1	2	2	1	2	2	3	2	2	4	2	3	4	5

Summing the entries of the bottom row, we conclude that $m = 37$.

Comment: When I set the advanced version of the quiz, I underestimated the answer. When I later solved the quiz, it did take me more than ten minutes.

Quiz 3: Let $G = A_5$, the alternating group of order 60. You may assume that the group algebra $\mathbb{C}G$ has exactly 5 simple modules, up to isomorphism, with dimensions 1, 3, 3, 4, 5. Up to isomorphism, how many simple 6-dimensional $\mathbb{C}G$-modules are there?

Solution: Write S_0, ..., S_4 for representatives of the isomorphism classes of simple $\mathbb{C}G$-modules, enumerated such that their dimensions are 1, 3, 3, 4, 5, respectively. Any $\mathbb{C}G$-module M is determined by the multiplicities m_0, ..., m_4, where $M \cong m_0 S_0 \oplus ... \oplus m_4 S_4$. Now supposing that $\dim(M) = 6$, then

$$6 = m_0 + 3m_1 + 3m_2 + 4m_3 + 5m_4 \; .$$

The number of possibilities for theisomorphism class of M is the number of natural number solutions $m_0...m_4 = (m_0, ..., m_4)$ to that equation. The solutions are

10001 , 20010 , 00200 , 02000 , 01100 , 30100 , 31000 , 60000 .

Thus, the answer is 8.

Quiz 4: The ordinary character table of the group $S_3 = \langle a, b : a^3 = b^2 = (ab)^2 \rangle$ is as shown. Evaluate the natural numbers λ, μ, ν where $(\chi_2)^2 = \lambda\chi_0 + \mu\chi_1 + \nu\chi_2$.

	1	3	2	$\lvert[g]\rvert$
	1	2	3	$\lvert\langle g\rangle\rvert$
$\chi(g)$	1	b	a	g
χ_0	1	1	1	
χ_1	1	-1	1	
χ_2	2	0	-1	

Solution: Let $\psi = (\chi_2)^2$. Now $(\psi(1), \psi(b), \psi(a)) = (4, 0, 1)$. By inspection, $\psi = \chi_0 + \chi_1 + \chi_2$. So $\lambda = \mu = \nu = 1$.

Comment: We could also directly calculate $\lambda = \langle \chi_0 \,|\, \psi \rangle$ and similarly for μ and ν.

Quiz 5: Let $H \leq G$ be finite groups and χ an irreducible $\mathbb{C}G$-character. Show that there exists an irreducible $\mathbb{C}H$-character ψ such that $\langle \chi \,|\, \mathrm{ind}_H^G(\psi) \rangle > 0$.

Solution: The regular $\mathbb{C}G$-character χ_{reg}^G is given by

$$\chi_{\mathrm{reg}}^G = \sum_{\chi \in \mathrm{Irr}(\mathbb{C}G)} \chi(1)\chi \; .$$

From the formula $\chi_{\mathrm{reg}}^G(g) = |G|\delta_{g,1}$, with $g \in G$, we see that $\chi_{\mathrm{reg}}^G = \mathrm{ind}_H^G(\chi_{\mathrm{reg}}^H)$. So

$$\sum_{\psi \in \mathrm{Irr}(\mathbb{C}H)} \psi(1) \langle \chi \,|\, \mathrm{ind}_H^G(\psi) \rangle = \langle \chi \,|\, \mathrm{ind}_H^G(\chi_{\mathrm{reg}}^H) \rangle = \langle \chi \,|\, \chi_{\mathrm{reg}}^G \rangle = \chi(1) \; .$$

It follows that $\langle \chi \,|\, \mathrm{ind}_H^G(\psi) \rangle > 0$ for some ψ.

Quiz 6: Consider the group $D_8 = \langle a, b \rangle$ where a is a rotation through a quarter of a revolution and b is a reflection. The character table of the subgroup $C_4 = \langle a \rangle$ is as follows.

$\chi(g)$	1	a	a^2	a^3	g
ϕ_0	1	1	1	1	
ϕ_1	1	i	-1	$-i$	
ϕ_2	1	-1	1	-1	
ϕ_3	1	$-i$	-1	i	

Fill in the entries of the following table of characters induced to D_8 from C_4.

	1	1	2	2	2	$\lvert[g]\rvert$
	1	2	4	2	2	$\lvert\langle g \rangle\rvert$
	1	a^2	a	b	ab	g
$\mathrm{ind}(\phi_0)$	?	?	?	?	?	
$\mathrm{ind}(\phi_1)$	?	?	?	?	?	
$\mathrm{ind}(\phi_2)$	?	?	?	?	?	
$\mathrm{ind}(\phi_3)$	?	?	?	?	?	

Solution: Using the formula for induced characters, we obtain the following table.

	1	a^2	a	b	ab	g
$\mathrm{ind}(\phi_0)$	2	2	2	0	0	
$\mathrm{ind}(\phi_1)$	2	-2	0	0	0	
$\mathrm{ind}(\phi_2)$	2	2	-2	0	0	
$\mathrm{ind}(\phi_3)$	2	-2	0	0	0	

Quiz 7: Let $V = \mathbb{R}^3$ as an $\mathbb{R}S_4$-module with S_4 transitively permuting the vertices of a regular tetrahedron in V. Enter, into the following table, the values of the $\mathbb{C}S_4$-character $\chi_{\mathbb{C}V}$ of the $\mathbb{C}S_4$-module $\mathbb{C}V = \mathbb{C} \otimes_{\mathbb{R}} V$.

	1^4	2.1^2	2^2	3.1	4	g
$\chi_{\mathbb{C}V}$	?	?	?	?	?	

Solution: We shall show that the entries are as follows.

	1^4	2.1^2	2^2	3.1	4	g
$\chi_{\mathbb{C}V}$	3	1	-1	0	-1	

The dimension of $\mathbb{C}V$ is $\chi_{\mathbb{C}V}(1) = 3$. The eigenvalues of each reflection 2.1^2 are 1, 1, -1, which sum to $\chi_{\mathbb{C}V}(2.1^2) = 1$. The eigenvalues of each rotation 2^2 are 1, -1, -1, which sum to $\chi_{\mathbb{C}V}(2^2) = -1$. The eigenvalues of each rotation 3.1 are 1, ω, ω^2, where $\omega = e^{2\pi i/3}$, hence $\chi_{\mathbb{C}V}(3.1) = 0$. The eigenvalues of the reflections with shape 4 are -1, i, $-i$, which sum to $\chi_{\mathbb{C}V}(4) = -1$.

Alternative solution: Let χ_0 denote the trivial $\mathbb{C}S_4$-character. The $\mathbb{C}S_4$-character $\chi_{\mathbb{C}V} + \chi_0$, being the $\mathbb{C}S_4$-character of the $\mathbb{C}S_4$-module associated with the natural S_4-set, has values 4, 2, 0, 1, 0 at 1^4, 2.1^2, 2^2, 3.1, 4, respectively.

Quiz 8: The group $\mathrm{SL}_2(3)$ is the group of $2{\times}2$ matrices over the field with order 3. We have a semidirect product $\mathrm{SL}_2(3) = C_3 \ltimes Q_8$. Let $\omega = e^{2\pi i/3}$. Write a for a generator of the subgroup C_3. Write $Q_8 = \{1, i, j, k, z, iz, jz, kz\}$ in the usual way. We saw in class that part of the character table for $\mathrm{SL}_2(3)$ is as follows. (The first 4 rows are inflated from the quotient group $A_4 \cong \mathrm{SL}_2(3)/\langle z \rangle$. The first entries of χ_4, χ_5, χ_6 rows come from column orthonormality. The second entries of those three rows come from column orthonormality together with the fact that the only possible eigenvalues of the action of z are ± 1.) Determine the entries labelled s, s', s'', t, t', t''.

	1	1	6	4	4	4	4	$\lvert[g]\rvert$
	1	2	4	3	3	6	6	$\lvert\langle g\rangle\rvert$
	1	z	i	a	a^2	az	a^2z	g
χ_0	1	1	1	1	1	1	1	
χ_1	1	1	1	ω	ω^2	ω	ω^2	
χ_2	1	1	1	ω^2	ω	ω^2	ω	
χ_3	3	3	-1	0	0	0	0	
χ_4	2	-2	s	t	?	?	?	
χ_5	2	-2	s'	t'	?	?	?	
χ_6	2	-2	s''	t''	?	?	?	

Solution: By column orthogonality, $|s|^2 + |s'|^2 + |s''|^2 = 0$. Therefore, $s = s' = s'' = 0$.

By column orthonormality, t and t' and t'' cannot all be 0. By considering tensor products with χ_1 and χ_2, we may assume that $t' = \omega t$ and $t'' = \omega^2 t$. Column orthonormality now gives $|t| = 1$. But t must also be the sum of two cube roots of unity. We deduce that, numbering χ_4, χ_5, χ_6 suitably, then $t = -1$ and $t' = -\omega$ and $t'' = -\omega^2$.

Comment: The rest of the character table can now be determined easily, and it is as follows.

	1	z	i	a	a^2	az	a^2z	g
χ_4	2	-2	0	-1	-1	1	1	
χ_5	2	-2	0	$-\omega$	$-\omega^2$	ω	ω^2	
χ_6	2	-2	0	$-\omega^2$	$-\omega$	ω^2	ω	

To see this, first note that, for the simple module S with character χ_4, the eigenvalues of the action of a must be ω and ω^2, both with multiplicity 1. The eigenvalues of the action of a^2 must be the same. Since z acts on S as negation, the eigenvalues of the action of az must be $-\omega$ and $-\omega^2$, with both multiplicities 1. A similar comment holds for a^2z. All the values for χ_4 are now clear. Using tensor products by χ_1 and χ_2 again, we obtain the remaining entries.

MATH 525: Group Representations

Midterm

13 December 2023, LJB

1: (25 marks.) Let K be a field of characteristic 0. Let G be a finite group. Let A be an algebra over K with a basis consiting of elements $e(g)$ where g runs over the elements of G. Suppose there is a function $\alpha : G \times G \to K - \{0\}$ such that $e(g)e(h) = \alpha(g,h)e(gh)$ for all $g, h \in G$. Show that A is semisimple.

2: (25 marks.) Let F be a field. As an algebra over F, let W be the quotient of the free algebra on X and Y by the ideal generated by $YX - XY - 1$. Show that W is simple but not semisimple.

3: (25 marks.) Find the ordinary character table of the group A_6. You may state, without proof, the character tables of smaller finite groups.

4: (25 marks.) Let $H \leq G$ be finite groups and χ a faithful $\mathbb{C}G$-character (meaning that, as a homomorphism with domain G, the representation associated with χ is injective). Show that H is abelian if and only if, for every irreducible $\mathbb{C}H$-character ψ satisfying $\langle \psi \,|\, \mathrm{res}^G_H(\chi) \rangle_H \neq 0$, we have $\psi(1) = 1$.

Solutions to Midterm

1: We generalize a standard proof of Maschke's Theorem, taking care over some complications that arise. Replacing $e(1)$ with $e(1)/\alpha(1,1)$, we reduce to the case where $e(1)$ is an idempotent. But $e(1)A = A$, so $e(1) = 1_A$.

We are to show that, given an A-module M with a submodule U, then there exists an A-submodule V of M satisfying $M = U \oplus V$. Let $\pi' : U \leftarrow M$ be any projection. We define $\pi : U \leftarrow M$ such that

$$\pi m = \frac{1}{|G|} \sum_{g \in G} e(g) \pi' e(g)^{-1} m$$

for $m \in M$. If $m \in U$, then each $e(g)m \in U$, hence $\pi' e(g)m = e(g)m$ and we deduce that $\pi m = m$. Therefore, π is a projection with image U.

We have $e(h)^{-1} e(g)^{-1} = \alpha(g,h)^{-1} e(gh)^{-1}$. So, for all $m \in M$ and $g \in G$, we have

$$e(g)\pi e(g)^{-1} m = \frac{1}{|G|} \sum_{h \in G} e(g)e(h)\pi' e(h)^{-1} e(g)^{-1} m = \frac{1}{|G|} \sum_{h \in G} e(gh)\pi' e((gh)^{-1}) m = \pi m \, .$$

Suppposing now that $m \in \ker(\pi)$, we deduce that each $e(g)^{-1} m \in \ker(\pi)$. Since $e(g)e(g^{-1}) = \alpha(g, g^{-1})e(1)$, each $e(g)^{-1}$ is a nonzero scalar multiple of $e(g^{-1})$. Therefore, each $e(g)m \in \ker(\pi)$. We have shown that $\ker(\pi)$ is an A-submodule of M. So we can put $V = \ker(\pi)$.

Comment: Such an algebra A is called a **twisted group algebra** over G. Using group cohomology, it can be shown that $A \cong KF\epsilon$ with the following notation. There is a short exact sequence of groups

$$1 \to E \to F \to G \to 1$$

where $E \leq Z(F)$ and E is an isomorphic copy of a finite subgroup of the multiplicative group $K - \{0\}$. Also, ϵ is a primitive idempotent of the commutative group algebra KE. The usual version of Maschke's Theorem already tells us that KF is semisimple. Since $KF = KF\epsilon \oplus KF(1-\epsilon)$, it follows that A is semisimple.

2: Abusing notation, the images of X and Y in W will also be written as X and Y. Any element a of W can be expressed as an F-linear combination of elements having the form $X^m Y^n$. When a is nonzero, we define the **degree** of x to be (m,n) where the coefficient of $X^m Y^n$ in a is nonzero, m is maximal and, subject to that, n is maximal. Thus, the degrees (m,n) are ordered lexicographically. Given nonzero elements a and a' of W with degrees (m,n) and (m', n'), respectively, then aa' has degree $(m+m', n+n')$. Therefore, the units of W are precisely the units of the subalgebra F. Yet W is infinite-dimensional over F. Therefore W cannot be a finite-dimensional matrix algebra over a division ring. In other words, W is not semisimple.

Let I be a nonzero ideal in W. Noting that $YX = XY + 1$, an inductive argument shows that $YX^m = nX^{m-1} + X^m Y$. Hence $Y(X^m Y^n) - (X^m Y^n)Y = mX^{m-1}Y^n$. Let a be a nonzero element of I with minimal degree (m,n). By considering the element $Ya - aY \in I$, we see that $m = 0$. By considering the element $Xa - aX$, we see that $n = 0$. We have shown that a is a nonzero element of F. Therefore $I = W$ and W is a simple algebra.

3: The ordinary character table of A_6 is as shown on the next page, where $\mu = (1+\sqrt{5})/2$ and $\nu = (1-\sqrt{5})/2$ It can be obtained using induction from the subgroups A_5 and S_4, the latter being embedded via the inclusion $S_4 \hookleftarrow \mathrm{Sym}\{1,2,3,4\} \times \mathrm{Sym}\{5,6\}$ given by $s \mapsto (s, t(s))$ where $t(s)$ is the transposition if and only if s has odd signature. We omit the details.

	1	45	40	40	90	72	72	$\|[g]\|$
	1^6	$2^2.1$	3.1^2	3^2	4.2	5.1_1	5.1_2	$\|\langle g\rangle\|$
χ_0	1	1	1	1	1	1	1	
χ_1	5	1	2	-1	-1	0	0	
χ_2	5	1	-1	2	-1	0	0	
χ_3	8	0	-1	-1	0	μ	ν	
χ_4	8	0	-1	-1	0	ν	μ	
χ_5	9	1	0	0	1	-1	-1	
χ_6	10	-2	1	1	0	0	0	

4: In one direction, the required conclusion is trivial. Conversely, suppose $\psi(1) = 1$ for every ψ appearing in the restriction $\mathrm{res}^G_H(\chi)$. The representation with domain G associated with χ is injective and must therefore restrict to an injective representation ρ with domain H. Bur ρ is a direct sum of 1-dimensional representations ρ_ψ with domain H. We have $1 = \ker(\rho) = \bigcap_\psi \ker(\rho_\psi)$, so H embeds in the direct product of the cyclic groups $H/\ker(\psi)$. We deduce that H is abelian.

MATH 525: Group Representations

Final

22 December 2023, LJB

The duration of the exam is 120 minutes. It is a closed book exam.

Please write your name on every sheet of paper that you submit.

1: (30 marks.) Four of the irreducible characters of the simple group with order 168 are as follows. Find the last two rows of the character table.

	1	21	56	42	24	24	$\lvert[g]\rvert$
	1	2	3	4	7	7	$\lvert\langle g\rangle\rvert$
χ_0	1	1	1	1	1	1	
χ_1	6	2	0	0	-1	-1	
χ_2	7	-1	1	-1	0	0	
χ_3	8	0	-1	0	1	1	

2: (30 marks.) The generalized quaternion group Q_{16} with order 16 is generated by elements a and b with relations $a^8 = 1$, $b^2 = a^4$, $bab^{-1} = a^{-1}$.

(a) Briefly, check that 1, a^4, a^2, b, ab, a, a^3 are representatives of the conjugacy classes.

(b) Find the ordinary character table of $\mathbb{C}Q_{16}$.

3: (20 marks.) Let G be a finite group and χ an irreducible $\mathbb{C}G$-character. By considering a formula the idempotent of $Z(\mathbb{C}G)$ corresponding to χ, show that $\chi(1)$ divides $|G|$.

4: (20 marks.) Two algebras A and B over a field are said to be **equivalent** provided there exist positive integers m and n and idempotents $e \in \mathrm{Mat}_m(A)$ and $f \in \mathrm{Mat}_n(B)$ such that $e\mathrm{Mat}_m(A)e \cong B$ and $f\mathrm{Mat}_n(B)f \cong A$. Let $G = S_7$, the symmetric group with degree 7. Up to equivalence, how many algebras are there having the form $\mathrm{End}_{\mathbb{C}G}(M)$ where M is a non-zero finite-dimensional $\mathbb{C}G$-module?

Solutions to Final

1: We define $z = (-1 + i\sqrt{7})/2 = \zeta + \zeta^2 + \zeta^4$ where $\zeta = e^{2\pi i/7}$. We shall show that the completion of the table is as follows.

	1	21	56	42	24	24	$\|[g]\|$
	1	2	3	4	7	7	$\|\langle g\rangle\|$
χ_4	3	-1	1	0	z	z^*	
χ_5	3	-1	1	0	z^*	z	

Let 1, g_2, g_3, g_4, a, b be representatives of the conjugacy classes, in the order of the columns. By orthonormality of the first column, $\chi_4(1)^2 + \chi_5(1)^2 = 168 - 1^2 - 6^2 - 7^2 - 8^2 = 18$. The only possibility is $\chi_4(1) = \chi_5(1) = 3$.

Consider the Sylow 7-subgroup $S = \langle a \rangle$. We have $|C_G(S)| = |C_G(a)| = 168/24 = 7$. So $C_G(S) = S$. But the number n of Sylow 7-subgroups of G is congruent to 1 modulo 7 and divides 24. Since G is simple, $n \neq 1$. Therefore, $n = 8$. It follows that $|N_G(S)| = 21$. So $N_G(S) \cong C_3 \ltimes S$, the unique non-abelian group with order 21. So the elements a and a^2 and a^4 are mutually G-conjugate.

Let $\chi_4(a) = \alpha$ and $\chi_5(b) = \beta$. Now α is a sum of three 7-th roots of unity, moreover, if a 7-th root of unity η is an eigenvalue of the action of g_5 on the simple module with character χ_4, then η^2 and η^4 are eigenvalues of that action. So the only possible values of α and β are 3 or z or z^*. If $\alpha = \beta = 3$, then the column orthonormality for $[a]$ fails. So at least one of α and β must be z or z^*. But z and z^* are non-real, so χ_4 and χ_5 must be complex conjugates and $\{\alpha, \beta\} = \{z, z^*\}$. Renumbering χ_4 and χ_5 if necessary, we may assume that $\alpha = z$ and $\beta = z^*$. Then $\chi_4(b) = z^*$ and $\chi_5(b) = z$. For $k \in \{2, 3, 4\}$, we have $\chi_4(k) = \chi_5(k)$, which can be evaluated using orthogonality with the first column.

Alternative: It can be shown (though the candidates were not expected to know it), that the group $G \cong \mathrm{GL}_3(2) \cong PSL_2(7)$ has an outer automorphism σ that interchanges the two conjugacy classes of elements with order 7. Using that fact, the following quicker argument becomes available. Since χ_0, χ_1, χ_2, χ_3 are constant on the elements with order 7, we may assume that $\chi_4(b) \neq \alpha$. Then σ must interchange χ_4 and χ_5, hence $\chi_4(b) = \beta$ and $\chi_5(b) = \alpha$. Then it is straightforward to determine α and β using column orthonormality.

2: Part (a). Noting that $ba^k = a^{-k}b$ for $k \in \mathbb{Z}$, we see that the conjugacy classes in Q_{16} are $\{b, a^2b, a^4b, a^6b\}$ and $\{ab, a^3b, a^5b, a^7b\}$ and those of the form $\{a^k, a^{-k}\}$.

Part (b). The character table is as shown.

	1	1	2	4	4	2	2	$\|[g]\|$
	1	2	4	4	4	8	8	$\|\langle g\rangle\|$
	1	a^4	a^2	b	ab	a	a^3	g
χ_0	1	1	1	1	1	1	1	
χ_1	1	1	1	-1	-1	1	1	
χ_2	1	1	1	1	-1	-1	-1	
χ_3	1	1	1	-1	1	-1	-1	
χ_4	2	2	-2	0	0	0	0	
χ_5	2	-2	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	
χ_6	2	-2	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	

The first 5 irreducible characters in the table are inflated from the quotient $Q_{16}/\langle a^4 \rangle \cong D_8$. The remaining characters χ_5 and χ_6 are induced from the faithful irreducible characters of the subgroup $\langle a \rangle \cong C_8$. An easy calculation of inner products confirms that χ_5 and χ_6 are irreducible.

3: For a contradiction, suppose that some prime p has higher multiplicity in $\chi(1)$ than in $|G|$. The primitive idempotent of $Z(\mathbb{C}G)$ associated with χ is

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g \ .$$

For any positive integer n, we have

$$e_\chi = e_\chi^{n+2} = \left(\frac{\chi(1)}{|G|} \right)^{n+2} \sum_g \lambda(g) g$$

for some algebraic integers $\lambda(g)$. Equating coefficients of the identity element, then multiplying by $|G|^{n+2}$, we obtain

$$\chi(1)^n \lambda(1) = |G|^{n+1} \ .$$

But this equation cannot hold when p^n does not divide $|G|$.

4: Given $\mathbb{C}G$-modules M and M', then $\mathrm{End}_{\mathbb{C}G}(M) \equiv \mathrm{End}_{\mathbb{C}G}(M')$ if and only if $\mathrm{End}_{\mathbb{C}G}(M)$ and $\mathrm{End}_{\mathbb{C}G}(M')$ have the same number of Wedderburn components. Letting M run over all the non-zero finite-dimensional $\mathbb{C}G$-modules, then the number $n(M)$ of Wedderburn components of $\mathrm{End}_{\mathbb{C}G}(M)$ is equal to the number of mutually non-isomorphic simple composition factors of M. So the number of possibilities for $n(M)$ is the number of simple $\mathbb{C}G$-modules up to isomorphism. That is equal to the number of conjugacy classes of G, in other words, the number of partitions of 7. Those partitions are

$$1^7,\ 2.1^5,\ 2^2.1^3,\ 2^3.1,\ 3.1^4,\ 3.2.1^2,\ 3.2^2,\ 3^2.1,\ 4.1^3,\ 4.2.1,\ 4.3,\ 5.1^2,\ 5.2,\ 6.1,\ 7 \ .$$

Therefore, the answer is 15.