Archive for MATH 523, Algebra 1, Fall 2020

Bilkent University, Laurence Barker, 24 January 2021.

Source file: arch523fall20.tex

MATH 523, Algebra I, Fall 2020

Course specification

Laurence Barker, Bilkent University. Version: 18 December 2020.

Course Description: The course is a treatment of core graduate algebra, largely group theory, ring theory and some introductory category theory.

Instructor: Laurence Barker, Office SAZ 129,
e-mail: barker at fen dot bilkent nokta edu dot tr.

Course Texts: The primary course text is:

I. M. Isaacs, "Algebra: a Graduate Course" (Brooks/Cole, Pacific Grove, 1994.)

A secondary text, more elementary and covering only some of the material, is:

D. S. Dummit, R. M. Foote, "Abstract Algebra", 3th Edition. (Wiley, 2004).

Course Documentation: On my homepage, see the files of **course notes** on selected topics, the file on **homeworks and solutions**.

Detailed Syllabus: The format below is *Week number: Monday date: Subtopics.*

1: 14 Sep: Review of abstract groups, permutation groups. Categories. Groups as automorphism groups for various categories.

2: 21 Sep: Subgroups, cosets, Lagrange's Theorem. Examples, such as dihedral groups, quaternion groups, symmetric groups, general linear groups.

3: 28 Sep: Permutation groups, Orbit-Stablizer Theorem. Symmetries of polyhedra. Centre, centralizers, normalizers. Conjugacy classes and the class formula.

4: 5 Oct: Normal subgroups, quotient groups. The Three Group Isomorphism Theorems. Direct and Semidirect Product Recognition Theorems. Automorphisms and inner automorphisms of groups.

5: 12 Oct: Finite p-groups and the Burnside Basis Theorem. Sylow Theorems.

6: 19 Oct: Functors. Universals. Illustrations from group theory, such as the abelianization of a group as a universal.

7: 26 Oct: Symmetric and alternating groups. Conjugacy classes. Simplicity of alternating groups.

8: 2 Nov: Structure Theorem for Finitely-Generated Abelian groups. Unit groups in modular arithmetic and automorphisms of cyclic groups.

9: 9 Nov: *Midterm Week.*

10: 16 Nov: Goursat's Theorem, Zassenhaus' Butterfly Theorem, Schreier Refinement Theorem, Jordan–Holder Theorem.

11: 23 Nov: Rings and ideals. Quotient rings. Algebras. Locally unital rings. Applications of Zorn's Lemma.

12: 30 Nov: The Three Isomorphism Theorems for Modules.

13: 7 Dec: Bimodules and tensor products.

14: 14 Dec: Equivalences of categories. Morita equivalences.

15: 21 Dec: Survey of functorial approach to group theory. Group functors and category algebras.

FZ Criterion: Very low homework and midterm marks, threshold to be determined by difficulty of the questions.

Assessment:

- Homework, 30%,
- Midterm, 30%,
- Final, 40%.

75% attendance is compulsory. Attendance will be assessed through electronic records. Exceptions will be made only for students with documentary evidence of health excemptions or course clashes.

MATH 523, Algebra I, Fall 2020

Homeworks and Solutions

Laurence Barker, Bilkent University. Version: 22 December 2020.

Homework A

This homework is to be submitted to Moodle by 12:00 noon, Monday, 5 October.

Exercise A.1: Evaluate the order of the group $\mathrm{Aut}(S_3)$. Determine the group $\mathrm{Aut}(S_3)$ up to isomorphism.

Optional Extra: Determine the group $\mathrm{Aut}(S_n)$ for other particular values of n, such as $n = 4$ or $n = 5$. A description of $\mathrm{Aut}(S_n)$ was obtained by Otto Hólder in 1895, and one-page proof of it was found by Irving Segal in a paper, *The automorphisms of the symmetric group*, Bulletin of the American Mathematical Society **46**, 565-565 (1940). Segel's proof, brief as it is, consists of technical calculations. I do not know whether there is a much nicer proof.

Exercise A.2: Let n be a positive integer. Evaluate the order of the group $\mathrm{Aut}(C_n)$. (Later, we shall describe how $\mathrm{Aut}(C_n)$ can be expressed, up to isomorphism, as a direct product of cyclic groups.)

The next two exercises have conclusions that will already seem inituitively very plausible. Composing clear proofs of them is a test of technique.

Exercise A.3: *Let Ω and Ω' be sets (not necessarily finite) such that $|\Omega| = |\Omega'|$ (as cardinal numbers). Show that $\mathrm{Sym}(\Omega) \cong \mathrm{Sym}(\Omega')$.*

Above, we saw that, up to isomorphism, there are exactly 1, 1, 1, 2 groups with orders 1, 2, 3, 4, respectively.

Exercise A.4: Let n be a positive integer. Show that, up to isomorphism, there are only finitely many groups with order n.

Exercise A.5: Let G be a non-trivial finite group and let $H \leq G$ such that $|G : H|$ is the smallest prime divisor of $|G|$. Show that $H \trianglelefteq G$.

Exercise A.6: Let G be a group and $E \leq F \leq G \geq H$. Show that $F \cap EH = E(H \cap F)$ (as subsets of G, not necessarily subgroups). This result is called Dedekind's Lemma.

Exercise A.7: Prove the Second and Third Isomorphism Theorems and the Direct Product Recognition Theorem.

Exercise A.8: Prove the Chinese Remainder Theorem in the following form: given coprime positive integers m and n, then $C_{mn} \cong C_m \times C_n$.

Exercise A.9: Up to isomorphism, how many groups of order 6 are there?

The following exercise may look pointless but, in fact, determining the subgroups of a group and determining automorphism groups of objects is often necessary in both theory and applications of group theory.

Exercise A.10: Find, up to isomorphism, all the groups of order 8. (Hint: there are exactly 5 of them.) For each of those groups, find all the subgroups. For each of those groups, evaluate the order of the automorphism group.

Homework B

This homework is to be submitted to Moodle by 12:00 noon, Wednesday, 4 November.

Exercise B.1: Let G be a group. Let $\rho : \mathrm{Sym}(G) \leftarrow G$ be the function such that, given $x, g \in G$, then $\rho(g)x = xg$. Given an example where ρ is not a permutation representation of G.

Exercise B.2: Let G_G denote G regarded as a G-set such that each $g \in G$ sends each $x \in G_G$ to xg^{-1}. Show that we have an isomorphism of G-sets ${}_GG \cong G_G$.

Exercise B.3: Show that C_3 is isomorphic to the automorphism group of a graph.

Exercise B.4: Show that every finite group is isomorphic to the automorphism group of a graph. (Hint: First modify the notion of a graph, allowing edges to have directions and colours. Define that notion clearly. Using that notion, consider a suitable case where the vertices are the group elements and the colours are the group elements too. Then find a way to get rid of the directions and colours.)

Exercise B.5: Regard the field of rational numbers $\mathbb{Q}$ as an abelian group under addition. Find subgroups $A \leq B \leq \mathbb{Q}$ such that $B/A \cong C_{p^\infty}$.

Exercise B.6: What is the smallest finite group G such that some mimimal generating set for G has more than the minimum number of generators?

Exercise B.7: Let P be a non-abelian group with order 8. How many minimal generating sets does P have?

Exercise B.8: Using Sylow's Theorem, show that there does not exist a simple group with order 30.

Exercise B.9: Let $\mathbb{F}$ be a finite field of prime characteristic p. Let $G = GL_n(\mathbb{F})$, the group of invertible $n \times n$ matrices over $\mathbb{F}$. Let S be the subset of G consisting of the upper unitriangular matrices, that is, the upper triangular matrices whose diagonal entries are all 1. Show that S is a Sylow p-subgroup of G.

Exercise B.10: Consider the following non-proof and non-theorem.

Non-theorem: Given a group G, then every strict subgroup of G is contained in a maximal subgroup of G.

Non-proof: Let $E < G$. Let $\mathcal{P}$ be the set of strict subgroups $F < G$ such that $E \leq F$. We partially order $\mathcal{P}$ by inclusion. Plainly, $\mathcal{P} \neq \emptyset$. Let $\mathcal{C}$ be a chain in $\mathcal{P}$. Then the unionset $\bigcup \mathcal{C}$ is an upper bound for $\mathcal{C}$ in $\mathcal{P}$. Therefore, by Zorn's Lemma, $\mathcal{P}$ has a maximal element. ⊠.

(1) Find a mistake in the non-proof.
(2) Find a counter-example to the non-theorem.
(3) Salvage something of the idea, by stating and proving a similar assertion.

Homework C

This homework is to be submitted to Moodle by Monday 14 December, by 12:00 noon.

Do the Practise Midterm below. I have set it up so that, if you wish to enjoy the adrenaline rush of an exam scene, you can first do it as a closed-book two-hour exam. After that, you could do it again in a more relaxed way, and submit just the second version to Moodle for marking.

MATH 523: Algebra I. **Practice Midterm.**

LJB, Fall 2020, Bilkent University.

Time allowed: two hours. Please put your name on EVERY sheet of your manuscript. The use of telephones, calculators or other electronic devices is prohibited. The use of very faint pencils is prohibited too. You may take the question sheet home.

1: 20 marks. Classify, up to isomorphism, the groups with order 12.

2: 20 marks. Let G be a group. Show that G is simple if and only if the subgroup $\{(g, g) : g \in G\}$ of $G \times G$ is maximal.

3: 20 marks Let F be a finite field with order $q = p^a$ where p is prime and a is a positive integer. Let n be an integer with $n \geq 2$. Let S be the p-group consisting of those upper triangular matrices over F whose diagaonal entries are all 1. Determine:
(a) the rank of S (that is, the minumum size of a generating set for S).
(b) the order $|\Phi(S)|$ of the Frattini subgroup $\Phi(S)$ of S.

4: 20 marks. Let G be a group and H a subgroup of G with index 2. Suppose that every element of $G - H$ has order 2. Show that H is abelian.

5: 20 marks. Let G be a group such that the automorphism group $\mathrm{Aut}(G)$ is trivial. What are the possible isomorphism classes of G? (Partial marks are avalable for dealing only with the case where G is finite.)

Solutions to Practice Midterm

Solution 1: Up to isomorphism, the groups with order 12 are:

$$C_4 \times C_3 \cong C_{12} \ , \quad V_4 \times C_3 \ , \quad A_4 \ , \quad E \ , \quad F$$

where $E = \langle u, v : u^4 = v^3, uvu^{-1} = v^{-1} \rangle$ and F is generated by a subgroup $V_4 = \{1, x, y, z\}$ and a subgroup $C_3 = \langle v \rangle$ with ${}^x v = {}^y v = v^{-1}$ and ${}^z v = v$. Plainly, those groups are of order 12 and mutually non-isomorphic. We must show that the list is complete.

Consider a group G with order 12. Let P be a Sylow 2-subgroup and Q a Sylow 3-subgroup. We have $Q \cong C_3$. Let v be a generator of Q.

First suppose $Q \lhd G$. If $P \cong C_4$, then $G \cong C_{12}$ or $G \cong E$ depending on whether or not the conjugation action P on Q is trivial. If $P \cong V_4$ then $G \cong V_4 \times C_3$ or $G \cong F$ depending on the same criterion.

Now suppose Q is not normal in G. By Sylow's Theorem, the number of G-conjugates of Q is 4. So G has exactly 8 elements with order 3. All the other non-trivial elements of G must belong to P. Therefore, $P \lhd G$. Since Q is not normal, the conjuation action of Q on P must be non-trivial. But C_4 has no automorphism with order 3. Hence $P \cong V_4$. There are exactly 2 automorphisms of V_4 with order 3. Those 2 automorphisms are inverses of each other. So there is only one possibility for the isomorphism class of G in this case, and we must have $G \cong A_4$.

Comment: In the notation of semidirect products, $E = C_4 \ltimes C_3$ and $F = V_4 \ltimes C_3$ where, in both cases, the action of the 2-group on the 3-group is the unique non-trivial action.

Solution 2: Write $\Delta(G) = \{(g, g) : g \in G\}$. The normal subgroups $H \trianglelefteq G$ and the intermediate subgroups $\Delta(G) \leq L$ are in a bijective correspondence $H \leftrightarrow L$ characterized by the condition $L = \Delta(G)(H \times H)$ and the equivalent condition that H is the kernel of the restriction $G \leftarrow L$ of the projection $G \ni g \hookleftarrow (g, g') \in G \times G$.

Solution 3: Recall, for any group E, the **derived subgroup** $D(E)$ is the subgroup of E generated by the commutators $[x, y] = xyx^{-1}y^{-1}$ with $x, y \in E$. Equivalently, $D(E)$ is the unique normal subgroup of E that is minimal subject to the condition that $E/D(E)$ is abelian.

Part (a). Given $s \in S$, write $s_{i,j}$ for the (i, j)-entry of s. For each $1 \leq k \leq n$, let S_k be the subgroup of S consisting of those elements s such that, $s_{i,j} = 0$ for all $1 \leq i < j \leq n$ satisfying $j - i < k$. We have a normal series

$$S = S_1 \rhd S_2 \rhd ... \rhd S_{n-1} \rhd S_n = 1 \ .$$

Now let $1 \leq k < n$. For $s, t \in S_k$ and $k < j$, we have

$$(st)_{j-k,j} = s_{j-k,j} + t_{j-k,j} \ .$$

Therefore, $(s^{-1})_{j-k,j} = -s_{j-k,j}$ and $(sts^{-1}t^{-1}) = s_{j-k,j} + t_{j-k,j} - s_{j-k,j} - t_{j-k,j} = 0$. It follows that $sts^{-1}t^{-1} \in S_{k+1}$. In particular, S_k/S_{k+1} is abelian.

Let $1 \leq i < j \leq n$ and $u, v \in F$. Write $u_{i,j}$ to denote the matrix with (i, j)-entry u and all other entries 0. For $i < h \leq n$, we have $(1 + u_{i,h})^{-1} = 1 - u_{i,h}$. If $i < h < j$, then

$$[1 + u_{i,h}, 1 + v_{h,j}] = (1 + u_{i,h} + v_{h,j} + (uv)_{i,j})(1 - u_{i,h} - v_{h,j} + (uv)_{i,j}) = 1 + (uv)_{i,j} \ .$$

Putting $v = 1$, we deduce that, if $2 \leq j - i$, then $1 + u_{i,j} \in D(S)$. Since $S_{n-1} = \{1 + u_{1,n} : u \in F\}$, we have $S_{n-1} \leq D(S)$. Inductively, for $2 \leq k \leq n-2$, suppose that $S_{k+1} \leq D(S)$. Since S_k is generated by S_{k+1} and the elements $u_{j-k,j} \in D(S)$ with $u \in F$ and $k < j \leq n$, we have $S_k \in D(S)$. We have shown that $S_2 \leq D(S)$. On the other hand, S/S_2 is abelian. Therefore $S_2 = D(S)$.

Let F_0 denote the minimal subfield of F. We mean to say, F_0 is the subfield of F with order p. Let $\mathcal{A}$ be an F_0 basis for F. Each element of S/S_2 can be written uniquely in the form $u(1)_{1,2} + u(2)_{2,3} + \ldots + u(n-1)_{n-1,n}$ with each $u(i) \in F$. Therefore, S/S_2 is elementary abelian and has an F_0-basis consising of the elements $u_{i,i+1}$ where $u \in \mathcal{A}$ and $1 \leq i \leq n-1$. Since $|\mathcal{A}| = a$, the rank of S/S_2 is $a(n-1)$. Since S/S_2 is elementary abelian, we have $S_2 \geq \Phi(S)$. But $\Phi(S) \geq D(S)$. Therefore $\Phi(S) = S_2$ and $\mathrm{rank}(S) = a(n-1)$.

Part (b). Each element s of S_2 is determined by the matrix entries $s_{i,j}$ with $2 \leq j - i$. There are $1 + 2 + \ldots + (n-1) + (n-2) = (n-1)(n-2)/2$ of those matrix entries. Therefore, $|\Phi(S)| = |S_2| = q^{(n-1)(n-2)/2}$.

Solution 4: Let $g \in G - H$. Given $h \in h$, then $g = g^{-1}$ and $1 = (gh)^2 = ghg^{-1}h$. That is to say, ${}^g h = h^{-1}$. Thus, conjugation by g effects an automorphism $h^{-1} \hookleftarrow h$ of H. Therefore, $h_1 h_2 = (h_2^{-1} h_1^{-1})^{-1} = (h_2^{-1})^{-1} (h_1^{-1})^{-1} = h_2 h_1$ for all $h_1, h_2 \in H$.

Solution 5: Since $\mathrm{Inn}(G)$ is trivial, G is abelian. Since the automorphism $g^{-1} \hookleftarrow g \in G$ must be the identity automorphism, every element of G must be of order 1 or 2. Therefore, G is the additive group of a vector space over the field $\mathbb{F}_2$ with order 2. Let $\mathcal{B}$ be an $\mathbb{F}_2$-basis for G. We have $|\mathcal{B}| \in \{0, 1\}$, because otherwise G would have a non-trivial automorphism interchanging 2 elements of $\mathcal{B}$. Therefore, G is the trivial group C_1 or the group C_2 with order 2.

Conversely, $\mathrm{Aut}(C_1) \cong \mathrm{Aut}(C_2) \cong C_1$. Therefore, the possible isomorphism classes for G are C_1 and C_2.

Comment: The argument relies on the fact the every vector space has a basis. Thus, we implicitly invoked the Axiom of Choice. I learned of this problem during a coffee discussion at a conference. One of the participants, struggling with the problem, was given the clue, "You need the Axiom of Choice". Amusingly, that clue seemed to supply scant help. Actually, though, I do not know whether the exclusion of infinite G really does need that assumption.

MATH 523: Algebra I. **Midterm.** LJB, 24 December 2020, Bilkent University.

Please write legibly and put your name on every sheet of your script.

1: 15 marks. Show that every group with order 40 is solvable.

2: 15 marks. Let E and F be finite groups. Show that $|E|$ and $|F|$ are coprime if and only if every subgroup of $E \times F$ has the form $A \times B$ where $A \leq E$ and $B \leq F$.

3: 30 marks. Classify the conjugacy classes of A_6, and find the size of each conjugacy class.

4: 40 marks. The group $\mathrm{SL}_2(3)$ is the group consisting of the 2×2 matrices with determinant 1 over the field $\mathbb{F}_3$ with order 3.

(a) Show that SL(2) has a unique element z with order 2. (Hint: use the formula for the inverse of a 2×2 matrix.)

(b) By considering the 1-dimensional subspaces of a 2-dimensional vector space over $\mathbb{F}_3$, show that $\mathrm{SL}_2(3)$ has a quotient group isomorphic to A_4.

(c) Using parts (a) and (b), find the number of Sylow 2-subgroups of $\mathrm{SL}_2(3)$ and determine the isomorphism class of the Sylow 2-subgroup.

(d) Using part (c), find the number of conjugacy classes of $\mathrm{SL}_2(3)$.

Solutions to MATH 523 Midterm, 24 December 2020.

Solution 1: Let G be a group with order 40. By Sylow's Theorem, the number n of Sylow 5-subgroups divides 8 and is congruent to 1 modulo 5. Plainly, $n = 1$. So G has a normal subgroup P with order 5. But $|G/P| = 8$, so G/P is solvable, hence G is solvable.

Solution 2: Suppose $|E|$ and $|F|$ are coprime. Write $\alpha : E \leftarrow E{\times}F$ and $\beta : F \leftarrow E{\times}F$ for the canonical projections. Given $h \in E{\times}F$ then, by the Chinese Remainder Theorem, $(\alpha(h), 1), (1, \beta(h)) \in \langle g \rangle$. So, given $H \leq E{\times}F$, we have $\alpha(H){\times}1, 1{\times}\beta(H) \leq H$. Therefore $H = \alpha(H){\times}\beta(H)$.

Conversely, suppose $|E|$ and $|F|$ have a common prime factor p. By Cauchy's Theorem, there exist elements $e \in E$ and $f \in F$ with order p. The subgroup $\langle (e, f) \rangle$ is not of the form $A{\times}B$.

Solution 3: For each of the integer partitions

$$1+1+1+1+1+1\,, \qquad 2+2+1+1\,, \qquad 3+1+1+1\,, \qquad 4+2\,, \qquad 3+3$$

there is a single conjugacy class of A_6 consisting of the elements of that shape. Those conjugacy classes have sizes 1, 45, 40, 90, 40, respectively. There are exactly 2 other conjugacy classes, both of them with size 72, which together consist of the elements with shape $5 + 1$.

To see this, first note that the 6 partitions above are precisely the partitions of 6 that have an even number of even terms. For the first 4 of those partitions, the corresponding elements are centralized by a transposition, and for the 5th, the corresponding elements are centralized by a product of 3 disjoint transpositions. Meanwhile, the normalizer of a 5-cycle g is generated by g and a 4-cycle, hence $C_G(g) \leq A_6$. It follows that all the partitions correspond to a single conjuacy class of A_6 except for $5 + 1$. The orders of the conjuacy classes are now clear.

Solution 4: Part (a). Let $S = \mathrm{SL}_2(3)$. Let $z \in S$. Write $z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $z^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ because $ad - bc = 1$. If $z = z^{-1}$, then $b = c = 0$ and $a = d$, whence $a^2 = 1$ and $a = \pm 1$. So z is an involution if and only if $a = d = -1$ and $b = c = 0$.

Part (b). The number of ordered basis of a 2-dimensional vector space over $\mathbb{F}_3$ is $|\mathrm{GL}_2(3)| = 8.6 = 48$. So $|S| = 24$. The action S permutes the four 1-dimensional subspaces of $\mathbb{F}_3^2$ and the intersection of the stabilizers of those subspaces is $Z(S) = \{1, z\}$. Therefore, $S/Z(S)$ is a subgroup of S_4 with order 12. It follows that $S/Z(S) \cong A_4$.

Part (c). Since A_4 has a normal Sylow 2-subgroup, it follows that S has a normal Sylow 2-subgroup. So the number of Sylow 2-subgroups of S is 1.

Part (d). By part (a), the Sylow 2-subgroup of S is isomorphic to Q_8. The elements of order 4 in Q_8 must be mutually S-conjugate. So there are precisely 3 conjugacy classes of 2-elements. Meanwhile, A_4 has precisely 2 conjuacy classes of elements with order 3. Therefore, S has precisely 2 conjugacy classes of 3-elements and 2 conjuacy classes of elements with order 6. In total, the number of conjuacy classes of S is 7.

MATH 523: Algebra I, Final.

LJB, 12 January 2021, Bilkent University.

Please write legibly and put your name on every sheet of your script.

1: 25 marks. Let L be a simple group with order 168. Without explicitly describing the isomorphism class of L, show that L is isomorphic to a subgroup of A_8. (Hint: use Sylow's Theorem.)

2: 25 marks. Let p be a prime and P a non-abelian group with order p^3. Find the number of conjugacy classes of P.

3: 25 marks. Let M and N be ideals of a unital ring R. Suppose that $M + N = R$. Show that there is a ring isomorphism $R/(M \cap N) \cong R/M \times R/N$. (Hint: write $1_R = m + n$ with $m \in M$ and $n \in N$.)

4: 25 marks. Construct functors as follows, confirming well-definedness and satisfaction of the definition of a functor.

(a) Given a group G, we define the **derived group** of G to be the normal subgroup generated by the comuutators $[x, y] = xyx^{-1}y^{-1}$, with $x, y \in G$. We define the **abelianization** of G to be the group $\mathrm{ab}(G) = G/D(G)$. By defining $\mathrm{ab}(\phi)$ for a group homomorphism ϕ, make ab() become a functor from the category of groups to the category of abelian groups.

(b) Let S and T be unital rings and let M be an S-T-bimodule. Given a T-module Y, let $f(Y) = M \otimes_T Y$ as an S-module. Make f become a functor by defining $f(\psi)$ for a map of T-modules ψ.

Solutions to MATH 523 Final, 12 January 2021

Solution 1: By Sylow's Theorem, the number n of Sylow 7-subgroups of L divides 24 and is congruent to 1 modulo 7. So $n = 8$. By considering the conjugation action of L on the set of Sylow 7-subgroups, we deduce that L is isomorphic to a subgroup L' of S_8. But $A_8 \cap L'$ is a non-trivial normal subgroup of L'. Since L' is simple, $L' \leq A_8$.

Solution 2: The centre $Z(P)$ is non-trivial and $P/Z(P)$ is non-cyclic, so $|Z(P)| = p$. For each $g \in P - Z(P)$, we have $C_P(g) = \langle g, Z(P) \rangle$. So $Z(P) < C_P(g) < P$. It follows that $|C_P(g)| = p^2$ and $|[g]_P| = p$. Therefore, the number of conjugacy classes of P is

$$|Z(P)| + |P - Z(P)|/p = p + (p^3 - p)/p = p^2 + p - 1 \ .$$

Solution 3: Let $\theta : R/M \times R/N \leftarrow R$ be the ring homomorphism such that $\theta(r) = (r+M, r+N)$ for $r \in R$. Then $\ker(\theta) = M \cap N$. So, by the Isomorphism Theorem, it suffices to show that θ is surjective. Let $r, s \in R$. We have $r + M = r(m + n) + M = rn + M$ and, similarly, $s + N = sm + N$. So $\theta(rn + sm) = (r + M, s + N)$. The required surjectivity is now clear.

Solution 4: Part (a). Let $\phi : F \leftarrow G$ be a group homomorphism. Then $\phi(D(G)) \leq D(F)$. So there is a well-defined group homomorphism $\mathrm{ab}(\phi) : \mathrm{ab}(F) \leftarrow \mathrm{ab}(G)$ given by $\mathrm{ab}(\phi)(gD(G)) = \phi(g)D(F)$. Plainly, ab() preserves identity morphisms and composites of morphisms. So ab() is a functor.

Part (b). Let $\psi : Y' \leftarrow Y$ be a T-map. Let A be the abelian group freely generated by the elements of $M \times X$. Let B be the subgroup of B generated by the elements having the form $(mt, y) - (m, ty)$ with $t \in T$. Then $f(Y) = A/B$. Defining A' and B' similarly for Y', then $f(Y') = A'/B'$. Let $\theta : A' \leftarrow A$ be the group homomorphism such that $\theta(m, y) = (m, \psi(y))$. Since $\psi(ty) = t(\psi(y))$, we have $\theta((mt, y) - (m, ty)) = (mt, \psi(y)) - (m, t\psi(y)) \in B'$. So $\theta(B) \leq B'$. Therefore, θ induces a homomorphism $f(\psi) : f(Y') \leftarrow f(Y)$. Plainly, f preserves identities and composites. So f is a functor.